ASSIGNMENT 4
Math 245 (Winter 2009)
Due at the start of class on Wednesday 4 February.

1. Let $T_1 \in \mathcal{L}(V_1)$ and $T_2 \in \mathcal{L}(V_2)$, where $V_1, V_2$ are vector spaces over a field $\mathbb{F}$. The spectrum of a linear transformation $T$ is $\text{spec}(T) := \{ \lambda \in \mathbb{F} : \lambda$ is an eigenvalue of $T \}$.

(a) Prove that if $\lambda \in \text{spec}(T_1)$ or $\lambda \in \text{spec}(T_2)$, then $\lambda \in \text{spec}(T_1 \oplus T_2)$.

(b) Prove or disprove: $\text{spec}(T_1 \oplus T_2) = \text{spec}(T_1) \cup \text{spec}(T_2)$.

(c) Prove that if $\lambda_1 \in \text{spec}(T_1)$ and $\lambda_2 \in \text{spec}(T_2)$, then $\lambda_1 \lambda_2 \in \text{spec}(T_1 \otimes T_2)$.

(d) Prove or disprove: $\text{spec}(T_1 \otimes T_2) = \{ \lambda_1 \lambda_2 : \lambda_1 \in \text{spec}(T_1) \text{ and } \lambda_2 \in \text{spec}(T_2) \}$.

2. Compute the eigenvalues and corresponding eigenspaces of the following matrices:

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -2 & -4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 \\ 8 & 25 & -16 \\ 14 & 43 & -27 \end{pmatrix}.$$

3. In this problem, all matrices are from $\mathcal{M}_{n \times n}(\mathbb{C})$ with $n \geq 2$.

(a) Compute the characteristic polynomial of

$$C_n := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & \vdots \\ \vdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

(b) What are the eigenvalues of $C_n$?

(c) Compute the eigenvalues of

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & a_1 & \cdots \\ \vdots & a_{n-2} & a_{n-1} & a_0 & \cdots & a_2 \\ a_2 & \cdots & \cdots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_0 \end{pmatrix}$$

for arbitrary $a_0, \ldots, a_{n-1} \in \mathbb{C}$. (Such a matrix is called a circulant matrix.)

4. Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$. We say that $T, U \in \mathcal{L}(V)$ are simultaneously diagonalizable when there is a single basis $\beta$ of $V$ such that $[T]_{\beta}, [U]_{\beta}$ are both diagonal. We say that $T, U$ commute when $TU = UT$.

(a) Prove that if $T, U$ are simultaneously diagonalizable, then they commute.
(b) Prove that if $T, U$ commute and each is diagonalizable, then $T, U$ are simultaneously diagonalizable.

(c) Show that the matrices

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -2 & -4 & 3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -2 & 2 \end{pmatrix}$$

commute, and find an invertible matrix $S$ such that $S^{-1}AS, S^{-1}DS$ are both diagonal.

5. An adjacency matrix is a square, symmetric zero-one matrix with all diagonal entries equal to 0. A permutation matrix is a square zero-one matrix with exactly one 1 in each row and in each column. Two adjacency matrices $A_1, A_2$ are said to be isomorphic when there exists some permutation matrix $P$ such that $A_2 = P^tA_1P$.

(a) Prove that if $A_1$ and $A_2$ are isomorphic, then $\text{spec}(A_1) = \text{spec}(A_2)$ (i.e., they have the same eigenvalues).

(b) Prove or disprove the converse of (a).