1. Let $V$ be a vector space, and let $S$ be a subset of $V$. The annihilator of $S$ is $S^0 := \{ f \in V^* : f(s) = 0 \text{ for all } s \in S \}$.

(a) Prove that $S^0$ is a subspace of $V^*$.

(b) Prove that $S^0 = \text{span}(S)^0$.

(c) Suppose $V$ is finite-dimensional, and $W$ is a subspace of $V$. What is $\text{dim}(W^0)$?

(d) Suppose $W$ is a subspace of $V$. Prove that $W^*$ is naturally isomorphic to $V^*/W^0$.

2. Let $T \in \mathcal{L}(V, W)$, where $V$ is a finite-dimensional vector space with inner product $\langle \cdot, \cdot \rangle_V$ and $W$ is a finite-dimensional vector space with inner product $\langle \cdot, \cdot \rangle_W$. We call $T^* \in \mathcal{L}(W, V)$ an adjoint of $T$ provided $\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V$ for all $v \in V$ and all $w \in W$. Prove the following:

(a) $T^*$ exists and is unique.

(b) If $\beta$ is an orthonormal basis for $V$ and $\gamma$ is an orthonormal basis for $W$, then $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$. 

(c) $\text{rank}(T^*) = \text{rank}(T)$.

(d) $\langle T^*(w), v \rangle_V = \langle w, T(v) \rangle_W$ for all $v \in V$ and all $w \in W$.

3. (a) Let $V, W$ be finite-dimensional inner product spaces, and let $T \in \mathcal{L}(V, W)$. Prove that $\mathcal{N}(T^*T) = \mathcal{N}(T)$.

(b) Let $A \in M_{n \times m}(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Deduce from part (a) that $\text{rank}(A^*A) = \text{rank}(A)$.

(c) Let $\langle \cdot, \cdot \rangle$ be an inner product over $\mathbb{F}^n$. The Gram matrix of a set of vectors $v_1, v_2, \ldots, v_m \in \mathbb{F}^n$ is the matrix $\Gamma \in M_{m \times m}(\mathbb{F})$ with entries $\Gamma_{jk} = \langle v_j, v_k \rangle$. Prove that $\text{rank}(\Gamma) = \text{dim}(\text{span}\{v_1, v_2, \ldots, v_m\})$.

4. Let $V$ be an inner product space over $\mathbb{C}$.

(a) Let $T \in \mathcal{L}(V)$ be normal. Let $p \in \mathbb{C}[x]$ be a polynomial with complex coefficients. Prove that $p(T)$ is normal.

(b) Let $U \in \mathcal{L}(V)$ be self-adjoint. Let $q \in \mathbb{R}[x]$ be a polynomial with real coefficients. Prove that $q(U)$ is self-adjoint.

(c) Does (b) remain true if $q$ has complex coefficients?

5. Let $x, y \in \mathbb{C}$, and let $B \in M_{3 \times 3}(\mathbb{C})$ be the matrix

$$B := \begin{pmatrix} x & y & y \\ y & x & y \\ y & y & x \end{pmatrix}.$$ 

(a) Show that $B$ is normal.

(b) Compute the eigenvalues of $B$, and find an orthonormal basis of eigenvectors.