ASSIGNMENT 7 Math 245 (Winter 2009)

Due at the start of class on Wednesday 11 March.

1. Let V be an n-dimensional inner product space over $\mathbb{F}$, and let $\mathsf{T} \in \mathcal{L}(\mathsf{V})$ be self-adjoint. We call T *positive semidefinite* if it satisfies $\langle \mathsf{T}(x), x\rangle \geq 0$ for all $x \in \mathsf{V}$. Prove that the following are equivalent:

   (i) T is positive semidefinite.

   (ii) The eigenvalues of T are non-negative.

   (iii) There exists a $\mathsf{U} \in \mathcal{L}(\mathsf{V})$ such that $\mathsf{T} = \mathsf{U}^*\mathsf{U}$.

   (Of course, we can make similar statements about matrices. If $A \in \mathsf{M}_{n\times n}(\mathbb{F})$ satisfies $A = A^*$, then we call A *positive semidefinite* if $x^*Ax \geq 0$ for all $x \in \mathsf{M}_{n\times 1}(\mathbb{F})$. While you are not asked to include the proof with your assignment, you are encouraged to convince yourself that A is positive semidefinite if and only if its eigenvalues are non-negative, if and only if there is a $B \in \mathsf{M}_{n\times n}(\mathbb{F})$ such that $A = B^*B$, if and only if L_A is positive semidefinite.)

2. The set of solutions $(x, y, z) \in \mathbb{R}^3$ to an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

   is called an *ellipsoid*. The volume of this ellipsoid is $\frac{4\pi}{3}abc$. The set of solutions to the equation

$$4(x^2 + y^2 + z^2) + 2(xy + xz + yz) = 1$$

   is also an ellipsoid. What is its volume? (Hint: You can make use of your solution to question 5 on assignment 6.)

3. Let n be a positive integer, and let $\omega := e^{2\pi i/n}$. The matrix

$$F := \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix} \in \mathsf{M}_{n\times n}(\mathbb{C})$$

   is called the *discrete Fourier transform modulo n*.

   (a) Show that F is unitary.

   (b) Recall from assignment 4 that a matrix of the form

$$C := \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & a_2 & & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & a_1 & \ddots & \vdots \\ \vdots & a_{n-2} & a_{n-1} & \ddots & \ddots & a_2 \\ a_2 & & \ddots & \ddots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_0 \end{pmatrix} \in \mathsf{M}_{n\times n}(\mathbb{C})$$

   is called a *circulant matrix*. Show that if C is a circulant matrix, then F^*CF is diagonal.

(c) Prove that if λ is an eigenvalue of F, then $\lambda \in \{1, -1, i, -i\}$.

4. Recall that if $A \in \mathsf{M}_{n\times n}(\mathbb{C})$, the exponential of A is defined as the matrix $e^A := \sum_{j=0}^{\infty} A^j/j!$.

   (a) Prove that if $A \in \mathsf{M}_{n\times n}(\mathbb{C})$ is Hermitian, then e^{iA} is unitary.

   (b) Prove that if $U \in \mathsf{M}_{n\times n}(\mathbb{C})$ is unitary, then there exists a Hermitian matrix A such that $U = e^{iA}$. Is A unique?

5. For any $\theta \in \mathbb{R}$, let

$$P_\theta := \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

   (a) Show that P_θ is an orthogonal projection matrix.

   (b) Let $\theta \in (0, \pi)$. Compute $\lim_{n\to\infty} (P_{n\theta} \cdots P_{2\theta} P_\theta)$.

   (c) Compute $\lim_{n\to\infty} (P_\theta \cdots P_{2\theta/n} P_{\theta/n})$ and compare your result with the previous part.

6. **(Bonus question)** If $\mathcal{E} \in \mathcal{L}(\mathcal{L}(\mathsf{V}))$ for some vector space V, then $\mathcal{E}$ is called a *superoperator*. In this problem, let $\mathsf{V} = \mathbb{C}^n$. By a slight abuse of notation, we will identify a linear transformation $\mathsf{T} \in \mathcal{L}(\mathsf{V})$ with its matrix $[\mathsf{T}]_\beta \in \mathsf{M}_{n\times n}(\mathbb{C})$ in a fixed orthonormal basis β, and thus we will call $\mathcal{E} \in \mathcal{L}(\mathsf{M}_{n\times n}(\mathbb{C}))$ a superoperator.

   (a) A superoperator $\mathcal{E}$ is called *positive* if, for all positive semidefinite matrices $A \in \mathsf{M}_{n\times n}(\mathbb{C})$, $\mathcal{E}(A)$ is positive semidefinite. Define $\mathcal{T} : \mathsf{M}_{n\times n}(\mathbb{C}) \to \mathsf{M}_{n\times n}(\mathbb{C})$ by $\mathcal{T}(A) = A^t$. Show that $\mathcal{T}$ is a positive superoperator.

   (b) A superoperator $\mathcal{E}$ is called *completely positive* if, for all positive integers m and all positive semidefinite matrices $A \in \mathsf{M}_{nm\times nm}(\mathbb{C})$, $(\mathcal{E} \otimes \mathcal{I})(A)$ is positive semidefinite, where $\mathcal{I}$ denotes the identity superoperator on $\mathsf{M}_{m\times m}(\mathbb{C})$. Show that $\mathcal{T}$ is *not* completely positive.

   (c) Given a set of matrices $E_1, \ldots, E_k \in \mathsf{M}_{n\times n}(\mathbb{C})$, we call

$$\mathcal{E}(A) = \sum_{i=1}^{k} E_i A E_i^*$$

   a *Kraus representation* for $\mathcal{E} : \mathsf{M}_{n\times n}(\mathbb{C}) \to \mathsf{M}_{n\times n}(\mathbb{C})$. Show that any $\mathcal{E}$ with a Kraus representation is a completely positive superoperator.

   (d) Prove the *Kraus Representation Theorem*: if $\mathcal{E}$ is a completely positive superoperator, then it has a Kraus representation.