1. Let \( V \) be an \( n \)-dimensional inner product space over \( \mathbb{F} \), and let \( T \in \mathcal{L}(V) \) be self-adjoint. We call \( T \) positive semidefinite if it satisfies \( \langle T(x), x \rangle \geq 0 \) for all \( x \in V \). Prove that the following are equivalent:

(i) \( T \) is positive semidefinite.

(ii) The eigenvalues of \( T \) are non-negative.

(iii) There exists a \( U \in \mathcal{L}(V) \) such that \( T = U^*U \).

(Of course, we can make similar statements about matrices. If \( A \in M_{n \times n}(\mathbb{F}) \) satisfies \( A = A^* \), then we call \( A \) positive semidefinite if \( x^*Ax \geq 0 \) for all \( x \in M_{n \times 1}(\mathbb{F}) \). While you are not asked to include the proof with your assignment, you are encouraged to convince yourself that \( A \) is positive semidefinite if and only if its eigenvalues are non-negative, if and only if there is a \( B \in M_{n \times n}(\mathbb{F}) \) such that \( A = B^*B \), if and only if \( L_A \) is positive semidefinite.)

2. The set of solutions \((x, y, z) \in \mathbb{R}^3\) to an equation of the form

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

is called an ellipsoid. The volume of this ellipsoid is \( \frac{4\pi}{3}abc \). The set of solutions to the equation

\[
4(x^2 + y^2 + z^2) + 2(xy + xz + yz) = 1
\]

is also an ellipsoid. What is its volume? (Hint: You can make use of your solution to question 5 on assignment 6.)

3. Let \( n \) be a positive integer, and let \( \omega := e^{2\pi i/n} \). The matrix

\[
F := \frac{1}{\sqrt{n}} \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2}
\end{pmatrix} \in M_{n \times n}(\mathbb{C})
\]

is called the discrete Fourier transform modulo \( n \).

(a) Show that \( F \) is unitary.

(b) Recall from assignment 4 that a matrix of the form

\[
C := \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\
a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} & \vdots \\
a_{n-2} & a_{n-1} & a_0 & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
a_2 & \cdots & \cdots & \cdots & a_0 & a_1 \\
a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_0
\end{pmatrix} \in M_{n \times n}(\mathbb{C})
\]

is called a circulant matrix. Show that if \( C \) is a circulant matrix, then \( F^*CF \) is diagonal.
(c) Prove that if \( \lambda \) is an eigenvalue of \( F \), then \( \lambda \in \{1, -1, i, -i\} \).

4. Recall that if \( A \in M_{n \times n}(\mathbb{C}) \), the exponential of \( A \) is defined as the matrix \( e^A := \sum_{j=0}^{\infty} \frac{A^j}{j!} \).

(a) Prove that if \( A \in M_{n \times n}(\mathbb{C}) \) is Hermitian, then \( e^{iA} \) is unitary.

(b) Prove that if \( U \in M_{n \times n}(\mathbb{C}) \) is unitary, then there exists a Hermitian matrix \( A \) such that \( U = e^{iA} \). Is \( A \) unique?

5. For any \( \theta \in \mathbb{R} \), let

\[
P_\theta := \begin{pmatrix}
\cos^2 \theta & \cos \theta \sin \theta \\
\cos \theta \sin \theta & \sin^2 \theta
\end{pmatrix}.
\]

(a) Show that \( P_\theta \) is an orthogonal projection matrix.

(b) Let \( \theta \in (0, \pi) \). Compute \( \lim_{n \to \infty} (P_{n\theta} \cdots P_{2\theta}P_\theta) \).

(c) Compute \( \lim_{n \to \infty} (P_\theta \cdots P_{2\theta/n}P_{\theta/n}) \) and compare your result with the previous part.

6. (Bonus question) If \( E \in \mathcal{L}(\mathcal{L}(V)) \) for some vector space \( V \), then \( E \) is called a superoperator. In this problem, let \( V = \mathbb{C}^n \). By a slight abuse of notation, we will identify a linear transformation \( T \in \mathcal{L}(V) \) with its matrix \( [T]_\beta \in M_{n \times n}(\mathbb{C}) \) in a fixed orthonormal basis \( \beta \), and thus we will call \( E \in \mathcal{L}(M_{n \times n}(\mathbb{C})) \) a superoperator.

(a) A superoperator \( E \) is called positive if, for all positive semidefinite matrices \( A \in M_{n \times n}(\mathbb{C}) \), \( E(A) \) is positive semidefinite. Define \( T : M_{n \times n}(\mathbb{C}) \to M_{n \times n}(\mathbb{C}) \) by \( T(A) = A^t \). Show that \( T \) is a positive superoperator.

(b) A superoperator \( E \) is called completely positive if, for all positive integers \( m \) and all positive semidefinite matrices \( A \in M_{mn \times mn}(\mathbb{C}) \), \( (E \otimes I)(A) \) is positive semidefinite, where \( I \) denotes the identity superoperator on \( M_{m \times m}(\mathbb{C}) \). Show that \( T \) is not completely positive.

(c) Given a set of matrices \( E_1, \ldots, E_k \in M_{n \times n}(\mathbb{C}) \), we call

\[
E(A) = \sum_{i=1}^{k} E_i A E_i^*
\]

a Kraus representation for \( E : M_{n \times n}(\mathbb{C}) \to M_{n \times n}(\mathbb{C}) \). Show that any \( E \) with a Kraus representation is a completely positive superoperator.

(d) Prove the Kraus Representation Theorem: if \( E \) is a completely positive superoperator, then it has a Kraus representation.