ASSIGNMENT 9 Math 245 (Winter 2009)

Due at the start of class on Wednesday 25 March.

1. Given a matrix $A \in \mathsf{M}_{n\times n}(\mathbb{C})$, a *polar decomposition* of A is a factorization $A = UP$, where $U \in \mathsf{M}_{n\times n}(\mathbb{C})$ is unitary and $P \in \mathsf{M}_{n\times n}(\mathbb{C})$ is positive semidefinite. Such a decomposition always exists (see Theorem 6.28 on page 411).
   (a) Prove or disprove: A is normal if and only if $UP = PU$ for any polar decomposition $A = UP$.
   (b) Give a necessary and sufficient condition for $A = UP$ (a polar decomposition of A) to be self-adjoint.
2. Let V, W be finite-dimensional inner product spaces, and let $\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$. Prove that $\mathsf{U} = \mathsf{T}^+$ if and only if $\mathsf{TUT} = \mathsf{T}$, $\mathsf{UTU} = \mathsf{U}$, and UT and TU are self-adjoint.
3. Let V, W be finite-dimensional inner product spaces. We call $\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$ an *isometry* if $\|\mathsf{T}(x)\| = \|x\|$ for all $x \in \mathsf{V}$.
   (a) Show that $\mathsf{T}^*\mathsf{T} \in \mathcal{L}(\mathsf{V})$ is the identity transformation. What is TT^*?
   (b) Show that $\mathsf{T}^+ = \mathsf{T}^*$.
4. In class, we saw that the least-squares fit to data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in \mathbb{R}^2$ by a line of the form $y = mx + b$ is given by

$$\begin{pmatrix} m \\ b \end{pmatrix} = A^+ \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \qquad \text{where} \qquad A := \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix}.$$

   (a) Compute the singular value decomposition of the matrix A.
   (b) Give closed-form expressions for m and b. (You may assume that the data points are not all collinear, since in that case the fit would be trivial.)
   (c) In the course of an experiment, you collect the following data:

x	y
1	-1.3
2	-0.1
3	4.8
4	5.7
5	11.5

   Find a linear least-squares fit to your data, and plot the results.