Ex: Solve the system of differential equations

$$i\frac{dx}{dt} = y \qquad i\frac{dy}{dt} = x$$

with initial condition $x(0) = 1$, $y(0) = 0$

In other words: $$i\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$i\frac{d}{dt} S^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = S^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S S^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

let $\begin{pmatrix} z \\ w \end{pmatrix} = S^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$. $\frac{d}{dt}\begin{pmatrix} z \\ w \end{pmatrix} = \frac{d}{dt} S^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$.

so we have $$i\frac{d}{dt}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}$$

$i\frac{d}{dt} z = z$, $i\frac{d}{dt} w = -w$ $\longrightarrow$ general solution: $z(t) = c_1 e^{-it}$, $w(t) = c_2 e^{it}$

Now $$\begin{pmatrix} x \\ y \end{pmatrix} = S\begin{pmatrix} z \\ w \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z+w \\ z-w \end{pmatrix}$$

so $x(t) = c_1 e^{-it} + c_2 e^{it}$
$y(t) = c_1 e^{-it} - c_2 e^{it}$

initial conditions: $1 = x(0) = c_1 + c_2$, $0 = y(0) = c_1 - c_2$ $\Longrightarrow$ $c_1 = c_2 = \frac{1}{2}$

(equal weight in both eigensp

i.e. $x(t) = \frac{1}{2}(e^{it} + e^{-it}) = \cos t$
$y(t) = -\frac{1}{2}(e^{it} - e^{-it}) = -i\sin t$.

So the coordinates rotate in a circle.