In fact, because $\otimes$ is bilinear, it does not matter what bases we use to define the tensor product.

Theorem Let V_1, V_2 be vector spaces with bases γ_1, γ_2, resp.
Let $V_1 \otimes V_2 = \text{span}\{v_1 \otimes v_2 : v_1 \in \gamma_1, v_2 \in \gamma_2\}$.
Let β_1, β_2 be bases for V_1, V_2, resp.
Then $\beta = \{v_1 \otimes v_2 : v_1 \in \beta_1, v_2 \in \beta_2\}$ is also a basis for $V_1 \otimes V_2$.
$\qquad \qquad \overset{\shortparallel}{\beta_1 \otimes \beta_2}$

Proof:

We must show that β spans $V_1 \otimes V_2$, and is linearly independent.

Span: Let $v \in V_1 \otimes V_2$.
Since $\gamma_1 \otimes \gamma_2$ is a basis for $V_1 \otimes V_2$, we have

$$v = \sum_{j,k} a_{jk}\, x_{1j} \otimes x_{2k}$$

for some a_{jk}, $j = 1, 2, \dots, \dim V_1$, $k = 1, 2, \dots, \dim V_2$
where $x_{1j} \in \gamma_1$, $x_{2k} \in \gamma_2$

Since β_1 is a basis for V_1 and β_2 is a basis for V_2,

$$x_{1j} = \sum_{\ell} b_{j\ell}\, y_{1\ell}, \quad x_{2k} = \sum_m c_{km}\, y_{2m}$$

for some $b_{j\ell}$, $\ell = 1, 2, \dots, \dim V_1$,
c_{km}, $m = 1, 2, \dots, \dim V_2$
where $y_{1\ell} \in \beta_1$, $y_{2m} \in \beta_2$

So
$$v = \sum_{j,k} a_{jk} \Big(\sum_\ell b_{j\ell}\, y_{1\ell}\Big) \otimes \Big(\sum_m c_{km}\, y_{2m}\Big)$$
$$= \sum_{j,k,\ell,m} a_{jk} b_{j\ell} c_{km}\, y_{1\ell} \otimes y_{2m} \qquad \text{(by bilinearity)}$$

i.e., $v \in \text{span}(\beta_1 \otimes \beta_2)$.

Independence: If V_1, V_2 are both finite, then
since $\dim(V_1 \otimes V_2) = |\gamma_1 \otimes \gamma_2|$
$\qquad = |\beta_1 \otimes \beta_2|$,
and the vectors in $\beta_1 \otimes \beta_2$ span $V_1 \otimes V_2$,
they must be linearly independent
(otherwise they would span a smaller space).

If V_1 or V_2 is infinite-dimensional, this argument does not work.

Instead: Suppose $\sum_{l,m} d_{lm}\, y_{1l} \otimes y_{2m} = 0$,

we want to show that $d_{lm} = 0$ for all l, m.

Change the basis from β_1 to γ_1, and β_2 to γ_2, writing $y_{1l} = \sum_j e_{lj} x_{1j}$, $y_{2m} = \sum_k f_{mk} x_{2k}$.

Note that this is the inverse of the basis change we did before:

$$\sum_j e_{lj} b_{jl'} = \delta_{l,l'} \qquad \sum_k f_{mk} c_{km'} = \delta_{m,m'}$$

So

$$\begin{aligned} 0 &= \sum_{l,m} d_{lm}\, y_{1l} \otimes y_{2m} \\ &= \sum_{l,m} d_{lm} \Big(\sum_j e_{lj} x_{1j}\Big) \otimes \Big(\sum_k f_{mk} x_{2k}\Big) \\ &= \sum_{j,k,l,m} d_{lm} e_{lj} f_{mk}\, x_{1j} \otimes x_{2k} \end{aligned}$$

and since the x's are a basis, we have

$$\sum_{l,m} d_{lm} e_{lj} f_{mk} = 0 \quad \text{for each } j,k.$$

so

$$\begin{aligned} 0 &= \sum_{l,m} d_{lm} e_{lj} b_{jl'} f_{mk} c_{km'} \quad \text{for any } j,k,l',m' \\ &= \sum_{j,k,l,m} d_{lm} e_{lj} b_{jl'} f_{mk} c_{km'} \quad \text{for any } l',m' \\ &= \sum_{l,m} d_{lm} \delta_{ll'} \delta_{mm'} \\ &= d_{l'm'} \end{aligned}$$

so in fact the vectors are linearly independent. $\square$