ASSIGNMENT 2

CO 481/CS 467/PHYS 467 (Winter 2010)

Due in class on Wednesday, February 3.

1. The Hadamard gate and qubit rotations

(a) [3 points] Suppose that \((n_x, n_y, n_z) \in \mathbb{R}^3\) is a unit vector and \(\theta \in \mathbb{R}\). Show that
\[
e^{-i \frac{\theta}{2} (n_x X + n_y Y + n_z Z)} = \cos \left( \frac{\theta}{2} \right) I - i \sin \left( \frac{\theta}{2} \right) (n_x X + n_y Y + n_z Z).
\]

(b) [2 points] Find a unit vector \((n_x, n_y, n_z) \in \mathbb{R}^3\) and numbers \(\phi, \theta \in \mathbb{R}\) so that
\[
H = e^{i \phi} e^{-i \frac{\theta}{2} (n_x X + n_y Y + n_z Z)},
\]
where \(H\) denotes the Hadamard gate. What does this mean in terms of the Bloch sphere?

(c) [3 points] Write the Hadamard gate as a product of rotations about the \(x\) and \(y\) axes. In particular, find \(\alpha, \beta, \gamma, \phi \in \mathbb{R}\) such that
\[
H = e^{i \phi} R_y(\gamma) R_x(\beta) R_y(\alpha).
\]

2. Circuit identities.

(a) [2 points] Show that the following circuit swaps two qubits:

(b) [1 point] Verify that \(HXH = Z\).

(c) [3 points] Verify the following circuit identity:

(d) [3 points] Verify the following circuit identity:

Give an interpretation of this identity.

3. Teleporting through a Hadamard gate.

(a) [1 point] Write the state \((I \otimes H)|\beta_{00}\rangle\) in the computational basis.

(b) [3 points] Suppose Alice has a qubit in the state \(|\psi\rangle\) and also, Alice and Bob share a copy of the state \((I \otimes H)|\beta_{00}\rangle\). If Alice measures her two qubits in the Bell basis, what are the probabilities of the four possible outcomes, and in each case, what is the post-measurement state for Bob?

(c) [2 points] Suppose Alice sends her measurement result to Bob. In each possible case, what operation should Bob perform in order to have the state \(H|\psi\rangle\)?
4. **Universality of gate sets.** Prove that each of the following gate sets either is or is not universal. You may use the fact that the set \{\text{CNOT}, H, T\} is universal.

(a) [1 point] \{H, T\}
(b) [2 points] \{\text{CNOT}, T\}
(c) [2 points] \{\text{CNOT}, H\}
(d) [3 points] \{cZ, K, T\}, where cZ denotes a controlled-Z gate and \( K = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right) \)
(e) [challenge problem] \{cT^2, H\}, where cT^2 denotes a controlled-T^2 gate
(f) [challenge problem] \{\text{CNOT}, H, T^2\}

5. **Asymptotic notation.** Indicate whether the following statements are true or false.

(a) [1 point] \( 10n^3 + 3n^2 \in O(n^4) \)
(b) [1 point] \( 10n^3 + 3n^2 \in \Omega(n^4) \)
(c) [1 point] \( 1/n^2 \in O(1/n) \)
(d) [1 point] \( (2^n)^2 \in O(2^n) \)
(e) [1 point] \( (2^n)^2 \in 2^{\Theta(n)} \)

6. **One-out-of-four search.** Let \( f : \{0,1\}^2 \to \{0,1\} \) be a black-box function taking the value 1 on exactly one input. The goal of the one-out-of-four search problem is to find the unique \((x_1, x_2) \in \{0,1\}^2 \) such that \( f(x_1, x_2) = 1 \).

(a) [1 point] Write the truth tables of the four possible functions \( f \).
(b) [2 points] How many classical queries are needed to solve one-out-of-four search?
(c) [4 points] Suppose \( f \) is given as a quantum black box \( U_f \) acting as

\[
|x_1, x_2, y\rangle \xrightarrow{U_f} |x_1, x_2, y \oplus f(x_1, x_2)\rangle.
\]

Determine the output of the following quantum circuit for each of the possible black-box functions \( f \):

\[
\begin{array}{c|c|c|c|c}
 & |0\rangle & |0\rangle & |1\rangle \\
\hline \text{H} & |\rangle & \text{H} & \text{H} \\
\hline U_f \\
\end{array}
\]

(d) [2 points] Show that the four possible outputs obtained in the previous part are pairwise orthogonal. What can you conclude about the quantum query complexity of one-out-of-four search?