ASSIGNMENT 1 CO 481/CS 467/PHYS 467 (Winter 2012)

Due in class on Thursday, January 12.

1. *Universality of reversible logic gates.*

   (a) [3 points] The CCCNOT (triple-controlled NOT) gate is a four-bit reversible gate that flips its fourth bit if and only if the first three bits are all in the state 1. Show how to implement a CCCNOT gate using Toffoli gates. You may use additional workspace as needed. You may assume that bits in the workspace start with a particular value, either 0 or 1, provided you return them to that value. For a bonus point, give a circuit that works regardless of the values of any bits of workspace.

   (b) [4 points] Show that a Toffoli gate cannot be implemented using any number of CNOT gates, with any amount of workspace. Hence the CNOT gate alone is not universal. (Hint: It may be helpful to think of the gates as performing arithmetic operations on integers mod 2.)

2. *Computing reversibly.*

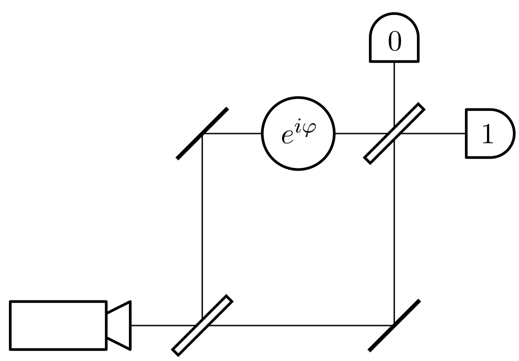
   The function EQ: $\{0,1\}^3 \to \{0,1\}$ determines whether its three input bits are equal, namely

$$\text{EQ}(x,y,z) = \begin{cases} 1 & \text{if } x = y = z \\ 0 & \text{otherwise.} \end{cases}$$

   (a) [2 points] Show how to compute the function EQ using AND, OR, NOT, and FANOUT gates.

   (b) [4 points] Show how to compute the function EQ reversibly using Toffoli gates. You may use ancilla bits initialized to either 0 or 1 provided you return them to that value. You may use gates other than Toffoli gates provided you explain how to implement any such gates using Toffoli gates.

3. *Mach-Zehnder interferometer with a phase shift.*



   Analyze the experiment depicted above using the mathematical model described in class. (Note that the model from class differs slightly from the model described in the textbook; in particular, you should use the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to model the beamsplitters.)

   (a) [4 points] Compute the quantum state of the system just before reaching the detectors. Express your answer using Dirac notation.

   (b) [2 points] Compute the probability that the "0" detector clicks as a function of φ, and plot your result for $\varphi \in [0, 2\pi]$.

4. *Pauli operators.*

   (a) [2 points] Express each of the three Pauli operators,

   $$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

   using Dirac notation in the computational basis.

   (b) [3 points] Find the eigenvalues and the corresponding eigenvectors of each Pauli operator. Express the eigenvectors using Dirac notation.

   (c) [2 points] Write the operator $X \otimes Z$ as a matrix and using Dirac notation (in both cases using the computational basis).

   (d) [2 points] What are the eigenspaces of the operator $X \otimes Z$? Express them using Dirac notation.

5. *Product and entangled states.* Determine which of the following states are entangled. If the state is not entangled, show how to write it as a tensor product; if it is entangled, prove this.

   (a) [2 points] $\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle$

   (b) [2 points] $\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)$

   (c) [2 points] $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$

6. *Unitary operations and measurements.* Consider the state

   $$|\psi\rangle = \frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle.$$

   (a) [2 points] Let $|\phi\rangle = (I \otimes H)|\psi\rangle$, where H denotes the Hadamard gate. Write $|\phi\rangle$ in the computational basis.

   (b) [3 points] Suppose the first qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the second qubit?

   (c) [3 points] Suppose the second qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the first qubit?

   (d) [2 points] Suppose $|\phi\rangle$ is measured in the computational basis. What are the probabilities of the four possible outcomes? Show that they are consistent with the marginal probabilities you computed in the previous two parts.