Appendix – Planar Structure Matching Under Projective Uncertainty for Geolocation

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Abstract. We enclose in appendix the complete proofs for the uncertainty modeling and line slope computation under projective transformations.

A Uncertainty Modeling

Lemma 1. *Let* $\mathbf{a}, \mathbf{b}$ *be column vectors in* $\mathbb{R}^n$ *and* $\|\mathbf{a}\| = 1$*, then*

$$\int_{t_1}^{t_2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\|\mathbf{a}t+\mathbf{b}\|^2}{2\sigma^2}} dt = e^{-\frac{\|\mathbf{b}\|^2 - (\mathbf{a}^\top \mathbf{b})^2}{2\sigma^2}} \cdot \frac{1}{2}\left(\mathit{erf}\left(\frac{t_2 + \mathbf{a}^\top \mathbf{b}}{\sqrt{2}\sigma}\right) - \mathit{erf}\left(\frac{t_1 + \mathbf{a}^\top \mathbf{b}}{\sqrt{2}\sigma}\right)\right) \tag{1}$$

Proof.

$$\int_{t_1}^{t_2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\|\mathbf{a}t+\mathbf{b}\|^2}{2\sigma^2}} dt \tag{2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{t_1}^{t_2} e^{-\frac{\|\mathbf{a}\|^2 t^2 + 2\mathbf{a}^\top \mathbf{b} t + \|\mathbf{b}\|^2}{2\sigma^2}} dt \tag{3}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{t_1}^{t_2} e^{-\frac{\left(\|\mathbf{a}\|t + \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\|}\right)^2 - \frac{(\mathbf{a}^\top \mathbf{b})^2}{\|\mathbf{a}\|^2} + \|\mathbf{b}\|^2}{2\sigma^2}} dt \tag{4}$$

$$= e^{-\frac{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}^\top \mathbf{b})^2}{2\sigma^2\|\mathbf{a}\|^2}} \int_{t_1}^{t_2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(t + \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\|^2}\right)^2}{2\sigma^2/\|\mathbf{a}\|^2}} dt \tag{5}$$

$$= e^{-\frac{\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}^\top \mathbf{b})^2}{2\sigma^2\|\mathbf{a}\|^2}} \cdot \frac{1}{2}\left(\mathrm{erf}\left(\frac{t_2 + \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\|^2}}{\sqrt{2}\sigma/\|\mathbf{a}\|}\right) - \mathrm{erf}\left(\frac{t_1 + \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\|^2}}{\sqrt{2}\sigma/\|\mathbf{a}\|}\right)\right) \tag{6}$$

Since $\|\mathbf{a}\| = 1$, hence

$$\int_{t_1}^{t_2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\|\mathbf{a}t+\mathbf{b}\|^2}{2\sigma^2}} dt = e^{-\frac{\|\mathbf{b}\|^2 - (\mathbf{a}^\top \mathbf{b})^2}{2\sigma^2}} \cdot \frac{1}{2}\left(\mathrm{erf}\left(\frac{t_2 + \mathbf{a}^\top \mathbf{b}}{\sqrt{2}\sigma}\right) - \mathrm{erf}\left(\frac{t_1 + \mathbf{a}^\top \mathbf{b}}{\sqrt{2}\sigma}\right)\right) \tag{7}$$

□

Theorem 1. *Let ℓ be a 2D line segment whose end points are random variables drawn from normal distributions $N(\mathbf{a}, \sigma^2)$ and $N(\mathbf{b}, \sigma^2)$ respectively. Then for any point $\mathbf{x}$, the probability that $\mathbf{x}$ lies on ℓ and ℓ has slope angle φ is*

$$p(\mathbf{x}, \varphi | \mathbf{a}, \mathbf{b}) = e^{-\frac{\|\mathbf{x}-\mathbf{a}\|^2 - |\langle \mathbf{x}-\mathbf{a}, \Delta_\varphi \rangle|^2 + \|\mathbf{x}-\mathbf{b}\|^2 - |\langle \mathbf{x}-\mathbf{b}, \Delta_\varphi \rangle|^2}{2\sigma^2}} \cdot \frac{1}{2}\left(1 - \mathit{erf}\left(\frac{\langle \mathbf{x}-\mathbf{a}, \Delta_\varphi \rangle}{\sqrt{2}\sigma}\right) \mathit{erf}\left(\frac{\langle \mathbf{x}-\mathbf{b}, \Delta_\varphi \rangle}{\sqrt{2}\sigma}\right)\right) \tag{8}$$

where $\Delta_\varphi = (\cos\varphi, \sin\varphi)^\top$ is the unit vector with respect to the slope angle φ.

Proof. Let $p_n(\mathbf{x}; \boldsymbol{\mu}, \sigma^2)$ be the probability density function for normal distribution $N(\boldsymbol{\mu}, \sigma^2)$, i.e.

$$p_n(\mathbf{x}; \boldsymbol{\mu}, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\|\mathbf{x}-\boldsymbol{\mu}\|^2}{2\sigma^2}} \tag{9}$$

The probability that $\mathbf{x}$ lies on the line segment equals the probability that random variables of the two ending points are $\mathbf{x} + t_a\Delta_\varphi$ and $\mathbf{x} + t_b\Delta_\varphi$ for some $t_a, t_b \in \mathbb{R}$ and $t_a \cdot t_b \leq 0$, therefore

$$\begin{aligned} p(\mathbf{x}, \varphi | \mathbf{a}, \mathbf{b}) = &\int_{-\infty}^{0} p_n(\mathbf{x} + t\Delta_\varphi; \mathbf{a}, \sigma^2) dt \int_{0}^{\infty} p_n(\mathbf{x} + t\Delta_\varphi; \mathbf{b}, \sigma^2) dt \\ &+ \int_{0}^{\infty} p_n(\mathbf{x} + t\Delta_\varphi; \mathbf{a}, \sigma^2) dt \int_{-\infty}^{0} p_n(\mathbf{x} + t\Delta_\varphi; \mathbf{b}, \sigma^2) dt \end{aligned} \tag{10}$$

According to Lemma 1,

$$\int_{-\infty}^{0} p_n(\mathbf{x} + t\Delta_\varphi; \mathbf{a}, \sigma^2) dt \int_{0}^{\infty} p_n(\mathbf{x} + t\Delta_\varphi; \mathbf{b}, \sigma^2) dt \tag{11}$$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\|\Delta_\varphi t + \mathbf{x} - \mathbf{a}\|^2}{2\sigma^2}} dt \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\|\Delta_\varphi t + \mathbf{x} - \mathbf{b}\|^2}{2\sigma^2}} dt \tag{12}$$

$$= e^{-\frac{\|\mathbf{x}-\mathbf{a}\|^2 - |\langle \Delta_\varphi, \mathbf{x}-\mathbf{a} \rangle|^2}{2\sigma^2}} \cdot \frac{1}{2}\left(\mathrm{erf}\left(\frac{\langle \Delta_\varphi, \mathbf{x}-\mathbf{a} \rangle}{\sqrt{2}\sigma}\right) + 1\right) \cdot e^{-\frac{\|\mathbf{x}-\mathbf{a}\|^2 - |\langle \Delta_\varphi, \mathbf{x}-\mathbf{b} \rangle|^2}{2\sigma^2}} \cdot \frac{1}{2}\left(1 - \mathrm{erf}\left(\frac{\langle \Delta_\varphi, \mathbf{x}-\mathbf{b} \rangle}{\sqrt{2}\sigma}\right)\right) \tag{13}$$

$$= \frac{1}{4} C \left(1 + \mathrm{erf}\left(\frac{\langle \Delta_\varphi, \mathbf{x}-\mathbf{a} \rangle}{\sqrt{2}\sigma}\right)\right)\left(1 - \mathrm{erf}\left(\frac{\langle \Delta_\varphi, \mathbf{x}-\mathbf{b} \rangle}{\sqrt{2}\sigma}\right)\right) \tag{14}$$

where $C = e^{-\frac{\|\mathbf{x}-\mathbf{a}\|^2 - |\langle \Delta_\varphi, \mathbf{x}-\mathbf{a} \rangle|^2 + \|\mathbf{x}-\mathbf{a}\|^2 - |\langle \Delta_\varphi, \mathbf{x}-\mathbf{b} \rangle|^2}{2\sigma^2}}$. Similarly, it can also be derived that

$$\int_{0}^{\infty} p_n(\mathbf{x} + t\Delta_\varphi; \mathbf{a}, \sigma^2) dt \int_{-\infty}^{0} p_n(\mathbf{x} + t\Delta_\varphi; \mathbf{b}, \sigma^2) dt \tag{15}$$

$$= \frac{1}{4} C \left(1 - \mathrm{erf}\left(\frac{\langle \Delta_\varphi, \mathbf{x}-\mathbf{a} \rangle}{\sqrt{2}\sigma}\right)\right)\left(1 + \mathrm{erf}\left(\frac{\langle \Delta_\varphi, \mathbf{x}-\mathbf{b} \rangle}{\sqrt{2}\sigma}\right)\right) \tag{16}$$

Therefore,

$$p(\mathbf{x},\varphi|\mathbf{a},\mathbf{b}) = \frac{1}{2}C\left(1-\mathrm{erf}\left(\frac{\langle \mathbf{x}-\mathbf{a},\Delta_\varphi\rangle}{\sqrt{2}\sigma}\right)\mathrm{erf}\left(\frac{\langle \mathbf{x}-\mathbf{b},\Delta_\varphi\rangle}{\sqrt{2}\sigma}\right)\right) \tag{17}$$

which is equivalent to Eq. 8. □

Proposition 1. *Let ℓ' be a line segment transformed from line segment ℓ in 2D space by nonsingular 3×3 projection matrix $\mathbf{P}$. If the two ending points of ℓ are random variables drawn from normal distributions $N(\mathbf{a},\sigma^2)$ and $N(\mathbf{b},\sigma^2)$ respectively, then for any $\mathbf{x}$, the probability that $\mathbf{x}$ lies on ℓ' and ℓ' has slope angle φ is*

$$p_{proj}(\mathbf{x},\varphi|\mathbf{P},\mathbf{a},\mathbf{b}) = p((x',\varphi') = proj(\mathbf{P}^{-1},\mathbf{x},\varphi)|\mathbf{a},\mathbf{b}) \tag{18}$$

where $proj(\mathbf{Q},\mathbf{x},\varphi)$ is a function returns the corresponding coordinate and slope angle with respect to $\mathbf{x}$ and φ after projection transformation $\mathbf{Q}$.

Proof. The mapping from $(\mathbf{x},\varphi)$ to $(\mathbf{x}',\varphi')$ is bijective, thus Eq. 18 holds. □

B Line Slopes Under Projective Transformation

The point coordinate transformed by $\mathbf{Q}$ can be obtained by homogeneous coordinate representation. For the slope angle, let $\mathbf{q}_i$ be the i-th row vector of projection matrix $\mathbf{Q}$, the transformed slope angle at location $\mathbf{x}=(x,y)^\top$ is φ'. Then $\tan\varphi' = \frac{dy'}{dx'}$ where

$$x'_{(x,y)} = \frac{\mathbf{q}_1^\top(x,y,1)^\top}{\mathbf{q}_3^\top(x,y,1)^\top} = \frac{q_{11}x+q_{12}y+q_{13}}{q_{31}x+q_{32}y+q_{33}} \tag{19}$$

$$y'_{(x,y)} = \frac{\mathbf{q}_2^\top(x,y,1)^\top}{\mathbf{q}_3^\top(x,y,1)^\top} = \frac{q_{21}x+q_{22}y+q_{23}}{q_{31}x+q_{32}y+q_{33}} \tag{20}$$

Since a line is still a line under projective transformation, hence

$$\frac{dy'}{dx'} = \frac{y'_{(x+\cos\varphi,y+\sin\varphi)} - y'_{(x-\cos\varphi,y-\sin\varphi)}}{x'_{(x+\cos\varphi,y+\sin\varphi)} - x'_{(x-\cos\varphi,y-\sin\varphi)}} \tag{21}$$

$$= \frac{\mathbf{q}_2^\top\mathbf{X}_+\mathbf{q}_3^\top\mathbf{X}_+ - \mathbf{q}_2^\top\mathbf{X}_-\mathbf{q}_3^\top\mathbf{X}_-}{\mathbf{q}_1^\top\mathbf{X}_+\mathbf{q}_3^\top\mathbf{X}_+ - \mathbf{q}_1^\top\mathbf{X}_-\mathbf{q}_3^\top\mathbf{X}_-} \tag{22}$$

where $\mathbf{X}_+ = (x+\cos\varphi, y+\sin\varphi,1)^\top$ and $\mathbf{X}_- = (x-\cos\varphi,y-\sin\varphi,1)^\top$. By equivalent transformations, it can be proved that $\mathbf{q}_2^\top\mathbf{X}_+\mathbf{q}_3^\top\mathbf{X}_+ - \mathbf{q}_2^\top\mathbf{X}_-\mathbf{q}_3^\top\mathbf{X}_- = f(\mathbf{q}_2,\mathbf{q}_3,x,y,\varphi)$ and $\mathbf{q}_1^\top\mathbf{X}_+\mathbf{q}_3^\top\mathbf{X}_+ - \mathbf{q}_1^\top\mathbf{X}_-\mathbf{q}_3^\top\mathbf{X}_- = f(\mathbf{q}_1,\mathbf{q}_3,x,y,\varphi)$ where

$$\begin{aligned} f(\mathbf{u},\mathbf{v},x,y,\varphi) =& (u_2v_1-u_1v_2)(x\sin\varphi - y\cos\varphi) \\ &+ (u_1v_3-u_3v_1)\cos\varphi + (u_2v_3-u_3v_2)\sin\varphi\ . \end{aligned} \tag{23}$$

Therefore,

$$\varphi' = \arctan\frac{f(\mathbf{q}_2,\mathbf{q}_3,x,y,\varphi)}{f(\mathbf{q}_1,\mathbf{q}_3,x,y,\varphi)} \tag{24}$$