Analysis of the Weight Function for Implicit Moving Least Squares Techniques

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Abstract

In this paper, we analyze the weight functions used in implicit moving least squares (IMLS) methods to construct implicit surfaces that interpolate or approximate polygon soup. We found that one method that presented an analytic solution to the integrated moving least squares method has issues with degeneracies because they changed the weight functions to decrease too slowly. We derive bounds for the choice of weight functions for IMLS methods to avoid these degeneracies for manifold data in two-dimensions and in three-dimensions.

Index Terms

Object modeling, surface reconstruction, implicit moving least squares, inverse distance weights.

1 INTRODUCTION

Implicit surfaces are a surface representation that have been applied in geometric modeling and surface reconstruction. Given unorganized 3D scattered data, reconstructing the implicit surface is of central importance in some computer graphics applications such as rendering. In the past decades, the problem of reconstructing implicit surfaces from scattered data has been explored extensively [Wen05]. The various approaches include signed distance estimation, Voronoi-based reconstruction, implicit surface fitting, and moving least squares surfaces [Cha08].

However, little work has looked at interpolating polygonal data. Shen et al. [SOS04] proposed a method of interpolating polygonal data which they called implicit moving least squares (IMLS). IMLS is a moving least squares technique to interpolate or approximate polygon soup (unorganized sets of polygons), requiring integration over each polygon. The Shen et al. method is based on the multi-level partition of unity (MPU) method of Ohtake et al. [OBA+03]. IMLS and MPU are fundamentally alike but the integrated constraints of Shen et al. are different from the collections of point constraints of Ohtake et al.

Our work was driven by analyzing the method of Shen et al. [SOS04]. Shen et al. derived their IMLS in a different way from the standard Moving Least Squares, which results in the squaring of the weight function. In our work, we found that the squaring of the weight function is necessary for its correctness for interpolating polygon soup (while we experimentally found this to be the case for polygon soup, in this paper we only prove the their weight functions needs to be squared for closed manifold data).

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Shen et al. were unable to derive a closed form of their integrals so they used numerical methods. Park et al. [PLK12] used the IMLS framework but with a different weight function. Their weight function allowed them to derive an analytic solution to the polygon integrals. However, based on tests we ran, we found that the weight function used by Park et al. gives incorrect implicit functions.

Motivated by the above, in this paper we derive a lower-bound of the rate of decrease of the IDW (inverse distance weight) function to guarantee that the implicit function is well-defined. We call an implicit function well-defined if and only if it is continuous, negative in a bounded region, and positive outside of a (possibly larger) bounded region. Based on this lower-bound, we provide criteria for choosing weight functions for both IMLS interpolation and approximation of closed polygons in 2D and polyhedron in 3D. In addition, for an IMLS function \( f \) we show regions of space in which \( f \) is positive and regions in which \( f \) is negative. Although IMLS schemes were devised as a way to interpolate and approximate non-manifold polyhedral data, in this paper our analysis focuses on manifold curves and surfaces.

The rest of this paper is organized as follows. In Section 2 we review Implicit Moving Least Squares, derived from the standard Moving Least Squares. We then present our constrains on the IMLS weight functions in Section 3. In Section 4 we summarize our results and discuss possible future research directions.

## 2 Implicit Moving Least Squares

Moving Least Squares (MLS) solves for every point \( x \) a locally weighted least squares problem. The influence of the data points is governed by a weight function \( w(x, y): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) which becomes smaller the further away its arguments are from each other. Ideally, \( w \) goes to zero as the distance between its arguments \( x, y \in \mathbb{R}^d \) goes to infinity. Such behavior can be modeled by using a translation-invariant weight function \( w(x - y) \). Wendland [Wen05] gives the following definition for moving least squares:

**Definition 2.1 (Moving Least Squares).** For a set of points \( P = \{p_1, \ldots, p_N\} \in \mathbb{R}^{d \times N} \), and a query point \( x \in \mathbb{R}^d \), the moving least squares approximation on point \( x \) is given by \( p^*(x) \) such that

\[
p^*(x) = \arg \min_p \left\{ \sum_{i=1}^{N} [p(x) - \phi_i]^2 w(x - p_i) : p \in \pi_M(\mathbb{R}^d) \right\},
\]

where \( \phi_i \) is the constraint on point \( p_i \) and \( \pi_M(\mathbb{R}^d) \) is the set of all polynomials of degree at most \( M \). The weight \( w(x) \) tends to zero as \( ||x|| \to \infty \).

According to Wendland [Wen05], the polynomial \( p \) can be defined in terms of basis functions \( b = [b_1, \ldots, b_M]^\top \in \mathbb{R}^M \) and adjusting their coefficients \( c = [c_1, \ldots, c_M]^\top \in \mathbb{R}^M \) as

\[
p = \sum_{j=1}^{M} c_j b_j.
\]

Given the basis function \( b(p_i) \) defined over sample \( p_i \), moving least squares (Eq. 1) is equivalent to minimizing over the coefficients

\[
R(c) = \sum_{i=1}^{N} \left[ \phi_i - \sum_{j=1}^{M} c_j b_j(p_i) \right]^2 w(x - p_i).
\]

Let \( B = [b(p_1), \ldots, b(p_N)]^\top \in \mathbb{R}^{N \times M} \) be the matrix of basis functions, \( W = \text{diag}(w(x - p_i) : i \in [0, N]) \) be the diagonal matrix of weight functions, and \( \phi = [\phi_1, \ldots, \phi_N]^\top \in \mathbb{R}^N \) be the collection of constraints, then Eq. 2 can be reformulated as

\[
R(c) = (\phi - Bc)^\top W(\phi - Bc).
\]
Because $R(c)$ is quadratic and $B^\top WB$ is positive semi-definite, the minimization problem of Eq. 3 has a unique solution [Wen05]. This minimum is found when $\nabla R(c) = 0$, i.e.,

$$\nabla R(c) = -2B^\top W\phi + 2B^\top WBc = 0 \iff B^\top WBc = B^\top W\phi,$$

(4)

the solution to which is $c^* = (B^\top WB)^{-1}B^\top W\phi$. The polynomial $p(x) = b^\top c^*$ is the solution to the moving least squares problem.

Shen et al. [SOS04] used $p$ as their solution to interpolate or approximate the data, renaming the function to $f$ (so $f(x) = p(x)$). Further, Shen et al. used constant basis functions (i.e., $b_j(p_i) = 1 \forall i, j$) and the MLS implicit function becomes

$$f(x) = \frac{\sum_{i=1}^{N} w(x - p_i)\phi_i}{\sum_{i=1}^{N} w(x - p_i)}.$$  

(5)

Shen et al. [SOS04] took a non-standard approach to MLS in their IMLS method (see [Yao14] for more details). To match their approach, we consider a generalized inverse distance weight function:

$$w_n(x - p_i) = \left(\frac{1}{\|x - p_i\|^2 + \epsilon^2}\right)^n,$$

(6)

where $n$ is a positive integer; Shen et al. only considered the case of $n = 2$. When $\epsilon = 0$, MLS interpolates the sample points (constraint values), and when $\epsilon \neq 0$, the MLS function approximates the constraint values. Greater values of $n$ assign greater influence to values closer to the sample point. In Section 3, we will prove that $n$ and $\epsilon$ have to meet certain conditions for IMLS to generate well-defined implicit curves and surfaces.

The MLS method for implicit surface reconstruction was originally applied to scattered point data approximation. However, the implicit surface function value for every sample point is $\phi_i = 0$. Directly using Eq. 5 is infeasible because it results in a trivial solution $f(x) = 0$. To apply MLS for surface reconstruction, the IMLS method in Shen et al. [SOS04] used a shape function $S(x)$ to incorporate normal constraints,

$$S_i(x) = \phi_i + (x - p_i)^\top n_i,$$  

(7)

where $x$ is the query point, $p_i$ is the sample point, $n_i$ is the normal at $p_i$, and $\phi_i$ is the implicit value of $p_i$, which is normally 0. Martin et al. [MKB+10] later used a similar shape function in their point-based moving least squares method.

When sampling finite points on a polygon for interpolation, the resulting implicit surface has bumps and dimples. To avoid this problem, Shen et al. integrate the constraints over each element (line segment in 2D or polygon in 3D) to achieve an infinite sampling rate. For a set of $K$ elements $\{\Omega_k : k = 1, \ldots, K\}$, the implicit function in Eq. 3 becomes

$$f(x) = \frac{\sum_{k=1}^{K} \int_{\Omega_k} w(x - p_{\Omega_k})S_k(x)dp_{\Omega_k}}{\sum_{k=1}^{K} \int_{\Omega_k} w(x - p_{\Omega_k})dp_{\Omega_k}},$$

(8)

where $p_{\Omega_k}$ is a point on $\Omega_k$ and $S_k(x) = \phi_k + (x - p_{\Omega_k})^\top n_{\Omega_k}$ is the shape function from Eq. 7. Since the elements are either line segments or polygons, the shape function $S_k(x)$ is constant for any $p_{\Omega_k}$ within the same element $\Omega_k$.

Shen et al. [SOS04] considered the weight function (Eq. 6) only for $n = 2$. They gave an analytic solution to their IMLS (Eq. 8) for 2D but only gave a numerical solution for the 3D case. Park et al. [PLK12] proposed an analytic solution for both 2D and 3D interpolation (i.e., $\epsilon = 0$). Their solution is limited to $n = 1$ for 2D and $n = 3/2$ for 3D. However, according to our experiments, Park et al.’s scheme produces degenerate implicit functions for both interpolation and approximation. When $\epsilon \neq 0$, their implicit function was negative.
everywhere (Fig. 1a). And when $\epsilon = 0$, although the resulting implicit function should only be zero at points on the input polygons, with their method the implicit values of all points on or outside the polygon are zero (Fig. 1b,c). The same problem occurs for their method in 3D, i.e., the implicit value of all points outside the data polyhedron is zero. While we have proven that Park et al.’s scheme is zero outside the input using their formulation of their method, we omit that proof here since it follows from the more general result derived in the next section; the interested reader is referred to [Yao14].

### 3 ANALYSIS OF THE IMLS WEIGHT FUNCTION

In this section, we will derive conditions to guarantee a well-defined implicit function in IMLS. Our analysis is based on a reformulation of IMLS implicit functions using 2D polar coordinates and 3D spherical coordinates. Integrating over polar directions in 2D and spherical directions in 3D on each segment, we will cast rays from the query point to intersect each ray with the polygonal curve/polyhedral surface, and use these intersections to compute the sign of the integrand in the direction of the ray. For our analysis, we only consider closed polygonal curves and polyhedral surfaces that do not intersect with themselves.

#### 3.1 Preliminaries

We present two lemmas that will be used in the proofs of both the 2D and 3D theorems. The basic idea of the lemmas is that the terms alternate in sign; if we pair up consecutive terms in the sum, then each pair sums to the same sign, and thus the entire sum is of that sign. See the appendix for the proofs.

**Lemma 3.1.** Let $N$ be an even positive integer and $r_1, \ldots, r_N \in \mathbb{R}$ be a monotonically increasing sequence, i.e., $r_i < r_j, \forall i < j$. Define

$$I = \sum_{i=1}^{N} (-1)^{i+1} \frac{r_i^t}{(r_i^2 + \epsilon^2)^n}. \tag{9}$$

where $t > 0$ and $\epsilon, n \in \mathbb{R}$. Then:

- If $n \leq t/2$, then $I \leq 0$, with $I = 0$ only when $\epsilon = 0$ and $n = t/2$;
- If $n > t/2$ and $r_N < \sqrt{t\epsilon^2/(2n-t)}$, then $I < 0$;
- If $n > t/2$ and $r_1 > \sqrt{t\epsilon^2/(2n-t)}$, then $I > 0$.
Lemma 3.2. Let \( N \) be an odd positive integer and \( r_1, \ldots, r_N \in \mathbb{R} \) be a monotonically increasing sequence, i.e. \( r_i < r_j, \forall i < j \). Define

\[
I = \sum_{i=1}^{N} (-1)^i \frac{r_i^t}{(r_i^2 + \epsilon^2)^n}
\]

where \( t > 0 \) and \( \epsilon, n \in \mathbb{R} \). When \( \epsilon = 0 \), \( I < 0 \). When \( \epsilon \neq 0 \):

- if \( n \leq t/2 \), then \( I < 0 \);
- if \( n > t/2 \) and \( r_N \leq \sqrt{te^2/(2n-t)} \), then \( I < 0 \);
- if \( n > t/2 \) and \( r_1 \geq \sqrt{te^2/(2n-t)} \), then \( I < 0 \).

3.2 Necessary Conditions on the Weight Function for IMLS in 2D

In this section, we derive an implicit moving least squares formulation that is equivalent to Eq. 8 by using 2D polar coordinates. With the polar coordinate formulation, we will show necessary conditions on the weight function for 2D IMLS interpolation and approximation to construct well-defined implicit functions. While these conditions are also sufficient conditions for 2D IMLS interpolation, they are not sufficient conditions for approximation; thus we also give some sufficient (but not necessary) conditions on the approximation method.

3.2.1 Polar Coordinate Formulation of IMLS in 2D

In 2D, with an input polygonal curve \( L = \{L_k\}, k = 1, \ldots, K \) and weight function \( w_n \) of Eq. 6, the implicit function in Eq. 8 is

\[
f(x) = \frac{\sum_{k=1}^{K} A_k}{\sum_{k=1}^{K} A_k},
\]

where \( A_k = \int_{L_k} w_n(x - p_{L_k}) dp_{L_k} \) and \( A_k = A_k S_k(x) \). The normal function is chosen with \( \phi = 0 \), i.e., \( S_k(x) = (x - p_{L_k})^T n_{L_k} \). Notice that \( S_k(x) \) is the same with respect to any point \( p_{L_k} \) on the same line segment. We transform all the points into polar coordinates \((r, \theta)\) where the origin is the query point \( x \), and \( \theta \) ranges from 0 to 2\( \pi \). For any point \( p_{L_k} \) on line \( L_k \),

\[
||x - p_{L_k}|| = r_k(\theta) = b_k \sec |\theta - \varphi|,
\]

where \( b_k \) is the distance from \( x \) to \( p_{L_k} \) and \( \varphi \) is the polar angle between \( L_k \) and its perpendicular line starting from \( x \). An illustration of the transformed polar coordinates is shown in Fig. 2. From Eq. 12 we know \( dr_k/\theta = b_k \sec |\theta - \varphi| \tan |\theta - \varphi| = r_k(\theta) \tan |\theta - \varphi| \) and \( S_k(x) = (x - p_{L_k})^T n_{L_k} = b_k \). Therefore,

\[
a_k = S_k(x) \int_{L_k} w_n(x - p_{L_k}) dp_{L_k} = b_k \int_{L_k} w_n(r_k) dp_{L_k}.
\]

By using the arc length formula for polar coordinates,

\[
a_k = b_k \int_{L_k} w_n(r_k) \sqrt{r_k^2 + \left( \frac{dr_k}{d\theta} \right)^2} d\theta
\]

\[
= b_k \int_{L_k} w_n(r_k) \sqrt{r_k^2 + r_k^2 \tan^2 |\theta - \varphi|} d\theta
\]

\[
= b_k \int_{L_k} w_n(r_k) \sqrt{r_k^2 \sec^2 |\theta - \varphi|} d\theta
\]

\[
= b_k \int_{L_k} w_n(r_k) \sqrt{r_k^2 (\frac{r_k}{b_k})^2} d\theta
\]

\[
= \text{sgn}(b_k) \int_{L_k} r_k^2 w_n(r_k) d\theta,
\]
Fig. 2. 2D polar coordinate notation. Green line is the input line segment and \( x \) is the query point.

where \( \text{sgn}(\cdot) \) is the sign function. We will use Eq. 14 in the next section to give necessary conditions on the weight function.

### 3.2.2 Necessary Condition on the Weight Function in 2D

The following theorem gives necessary conditions on the weight function in Eq. 6 for 2D IMLS.

**Theorem 3.3.** In 2D, for the implicit moving least squares method with an input manifold polygonal curve defined by the set of line segments \( L = \{L_k\} \), \( k = 1, \ldots, K \), \( p_{L_k} \) being a point on line segment \( L_k \), and \( x \) being the point of evaluation, let the implicit function be

\[
f(x) = \frac{\sum_{k=1}^{K} a_k}{\sum_{k=1}^{K} A_k}, \quad A_k = \int_{L_k} w(x - p_{L_k}) d p_{L_k}, \quad a_k = A_k S_k(x),
\]

with weight function \( w(x - p_{L_k}) = \left( \frac{1}{\|x - p_{L_k}\|^2 + \epsilon^2} \right)^n \) for an arbitrary \( n \in \mathbb{R} \), and normal function \( S_k(x) = (x - p_{L_k})^T n_{L_k} \). Then

- when \( n < 1 \) or \( n = 1 \) and \( \epsilon \neq 0 \), for any \( x \), \( f(x) < 0 \);
- when \( n = 1 \) and \( \epsilon = 0 \):
  - if \( x \) is in the exterior of \( L \) or lies on \( L \), then \( f(x) = 0 \);
  - if \( x \) is in the interior of \( L \), then \( f(x) < 0 \);
- when \( n > 1 \), for any \( x \), let \( r_{\min}(x) \) and \( r_{\max}(x) \) be the minimum and maximum distance from \( x \) to \( L \):
  - if \( r_{\max}(x) < \sqrt{\epsilon^2/(n-1)} \) then \( f(x) < 0 \).
  - if \( x \) is in the exterior of \( L \) and \( r_{\min}(x) > \sqrt{\epsilon^2/(n-1)} \), then \( f(x) > 0 \);
  - if \( x \) is in the interior of \( L \), and \( r_{\min}(x) \geq \sqrt{\epsilon^2/(n-1)} \), then \( f(x) < 0 \);

**Proof:** Since \( A_k = \int_{L_k} w_n(x - p_{L_k}) d p_{L_k} \) is always positive, the sign of the implicit function \( f(x) \) is determined by the numerator \( \sum_{k=1}^{K} a_k \). From Eq. 14

\[
\sum_{k=1}^{K} a_k = \sum_{k=1}^{K} \text{sgn}(b_k) \int_{L_k} r_k^2 w_n(r_k) d\theta = \sum_{k=1}^{K} \text{sgn}(b_k) \int_{L_k} \frac{r_k^2}{(r_k^2 + \epsilon^2)^n} d\theta.
\]

Eq. 15 integrates over each line segment and sums the integrations. A ray cast from \( x \) intersects \( L \) in several places. Rotating the ray 360° around \( x \), the intersections cover all the points on \( L \). This is illustrated in Fig. 3.
Fig. 3. Left: integrating over segments; Right: integrating over angle.

Therefore, for any \( \theta \), \( r_k^2w(r_k) \) can be computed by casting a ray \( \vec{r} \) from \( x \) in direction \( \theta \), and intersecting \( \vec{r} \) with \( L \). Let \( \ell_i \) be the \( i \)th intersection as we step along the ray from \( x \), where the value of \( \ell_i \) is the index of the line segment intersected. The set of intersections for a ray in direction \( \theta \) from \( x \) can be represent as \( R(\theta) = \{ \ell_i \} \), \( i = 1, 2, \ldots, N \), where \( N \) depends on \( \theta \). Therefore, the sum of \( a_k \)'s can be represented in another form:

\[
\sum_{k=1}^{K} a_k = \int \sum_{\ell \in R(\theta)} \text{sgn}(b_\ell) \frac{r_\ell^2}{(r_\ell^2 + \epsilon^2)^n} d\theta = \int \sum_{\ell \in R(\theta)} I(\theta) d\theta. \tag{16}
\]

As we step along the ray from \( x \), the elements of \( R(\theta) \) are in ascending order in their distance from \( x \), i.e., \( r_{\ell_i}(\theta) < r_{\ell_j}(\theta) \) for \( i < j \), as shown in Fig. 4a.

From the Jordan curve theorem, space is divided by the polygon into two components, the bounded interior and the unbounded exterior. Thus every line segment indexed by \( R(\theta) \) separates the ray into two components. There are a finite number of intersections located exactly on the shared end points of two line segments. If the two line segments are on the same side of the ray, the intersection is ignored. When the ray coincides with a line segment, the implicit value contribution of the line segment is always zero. See Fig. 4b for examples.

Discarding these intersections results in the ray always alternating between the exterior and interior of the polygon, and will not change the integral of Eq. 16. With these vertex/edge intersections excluded, we know from standard ray casting [Shi62] that if \( x \) is in the exterior of \( L \), then \( N \) is even, and if \( x \) is in the interior of \( L \), then \( N \) is odd.

When \( x \) is in the exterior of \( L \), we know that \( \text{sgn}(b_\ell) = (-1)^{j+1} \) where \( k \in R(\theta) \). Therefore, we can rewrite the integrand in Eq. 16 as

\[
I(\theta) = \sum_{j=1}^{N} (-1)^{j+1} \frac{r_{\ell_j}^2}{(r_{\ell_j}^2 + \epsilon^2)^n} \tag{17}
\]

where \( \ell_j \in R(\theta) \) and \( N \) is even. From Lemma 3.1 with \( t = 2 \)

- If \( n \leq 1 \), or \( n = 1 \) and \( \epsilon \neq 0 \) then \( I(\theta) < 0 \) for each \( \theta \), thus \( f(x) < 0 \);
- If \( n = 1 \) and \( \epsilon = 0 \) then \( I(\theta) = \sum_{j=1}^{n} (-1)^{j+1} \), which is 0 since \( N \) is even, thus \( f(x) = 0 \).
- If \( n > 1 \) and \( r_{\ell_N}(\theta) < \sqrt{\epsilon^2/(n-1)} \) then \( I(\theta) < 0 \). Let \( r_{max} = \max\{r_{\ell_N}(\theta)\} \) for all \( \theta \). If \( r_{max} < \sqrt{\epsilon^2/(n-1)} \) then \( I(\theta) < 0 \) for each \( \theta \), thus \( f(x) < 0 \).
Fig. 4. (a) Visual illustration of intersections for the ray cast from the point \( x \) to the polygon in direction \( \theta \). \( R(\theta) = \{\ell_1, \ell_2\} \) where \( \ell_1 = 1 \) and \( \ell_2 = 10 \). \( r_{\ell_1} \) and \( r_{\ell_2} \) are distances from \( x \) to intersections on 1st and 10th line segments. (b) Examples of discarded intersections. Black dot is the query point, red dots are discarded intersections, and green dots are valid intersections.

- If \( n > 1 \) and \( r_{\ell_1}(\theta) > \sqrt{\epsilon^2/(n-1)} \) then \( I(\theta) > 0 \). Let \( r_{\min} = \min\{r_{\ell_1}(\theta)\} \) for all \( \theta \). If \( r_{\min} > \sqrt{\epsilon^2/(n-1)} \) then \( I(\theta) > 0 \) for each \( \theta \), thus \( f(x) > 0 \).

When \( x \) is in the interior of \( L \), \( \text{sgn}(b_{\ell_j}) = (-1)^j \), the integrand in Eq. 16 becomes

\[
I(\theta) = \sum_{j=1}^{N} (-1)^j \frac{r_{\ell_j}^2}{(r_{\ell_j}^2 + \epsilon^2)^n}
\]

where \( \ell_j \in R(\theta) \) and \( N \) is odd. From Lemma 3.2 with \( \epsilon \neq 0 \) and \( t = 2 \),

- if \( n \leq 1 \) then \( I(\theta) < 0 \) for each \( \theta \), thus \( f(x) < 0 \).
- If \( n > 1 \) and \( r_{\ell_N}(\theta) \leq \sqrt{\epsilon^2/(n-1)} \) then \( I(\theta) < 0 \). We know \( r_{\max} = \max\{r_{\ell_N}(\theta)\} \) for all \( \theta \).
  - If \( r_{\max} \leq \sqrt{\epsilon^2/(n-1)} \) then \( I(\theta) < 0 \) for each \( \theta \), thus \( f(x) < 0 \).
- If \( n > 1 \) and \( r_{\ell_1}(\theta) \geq \sqrt{\epsilon^2/(n-1)} \) then \( I(\theta) < 0 \). We know \( r_{\min} = \min\{r_{\ell_1}(\theta)\} \) for all \( \theta \).
  - If \( r_{\min} \geq \sqrt{\epsilon^2/(n-1)} \) then \( I(\theta) < 0 \) for each \( \theta \), thus \( f(x) < 0 \).

When \( x \) is in the interior of \( L \), \( N \) is odd and \( \epsilon = 0 \), from Lemma 3.2 we have \( I(\theta) < 0 \), so \( f(x) \) is negative.

For \( x \) on \( L \), \( x \) is on a particular line segment \( L_{k'} \). The intersection of the ray \( \vec{r} \) with \( L_{k'} \) (which is actually \( x \)) has no contribution to \( \sum_{k=1}^{K} a_k \) because \( b_{k'} = 0 \) and \( a_{k'} = 0 \). Therefore, for a ray cast from \( x \) on \( L_{k'} \), the first intersection on \( L_{k'} \) is ignored. When the ray is in the direction of the outside (inside) of \( L \), it is the same as when \( x \) is in the exterior (interior) of \( L \). Therefore, for \( x \) on \( L \), the sign of \( f(x) \) is determined only in those cases when the interior and exterior have the same result, i.e., when \( n \leq 1 \), \( f(x) < 0 \); when \( n > 1 \) and \( r_{\max} < \sqrt{\epsilon^2/(n-1)} \), \( f(x) < 0 \).

What Theorem 3.3 tells us is that for \( n > 1 \), there is an unbounded region outside of \( L \) where \( f \) is strictly positive and when certain conditions are met, there is a bounded region inside \( L \) where \( f \) is strictly negative as illustrated in Fig. 5. This leads to the following corollaries.
Fig. 5. Positive and negative zones for IMLS approximation when $n > 1$. The solid circles are the input data. The minimum and maximum distances from the dashed circles to the solid circles are marked as $r_{\text{min}}$ and $r_{\text{max}}$ and $r_{\text{min}} = r_{\text{max}} = \sqrt{\epsilon^2/(n-1)}$.

**Corollary 3.4.** When using IMLS to approximate a manifold polygon ($\epsilon \neq 0$), if $n > 1$ and if there exists a point $x$ inside of $L$ such that $r_{\text{min}} \geq \sqrt{\epsilon^2/(n-1)}$ or $r_{\text{max}} \leq \sqrt{\epsilon^2/(n-1)}$, then the resulting implicit function $f$ is well-defined.

**Corollary 3.5.** When using IMLS to approximate a convex polygon ($\epsilon \neq 0$), if $n > 1$ then the resulting implicit function $f$ is well-defined. In addition the interior of $L$ is on the interior of $f$.

**Proof:** The result follows from Corollary 3.4 and by observing that when $x$ is on the interior, a ray cast from $x$ hits the polygon in exactly one location, which is negative.

The case when $\epsilon = 0$ is of particular interest, since it results in interpolation.

**Corollary 3.6.** In 2D, for the implicit moving least squares method interpolating a manifold polygon defined by the set of line segments $L = \{L_k\}, k = 1, \ldots, K$, $p_{L_k}$ being a point on line segment $L_k$, and $x$ being the point of evaluation, let the implicit function be $f(x) = \frac{\sum_{k=1}^{K} a_k}{\sum_{k=1}^{K} A_k}$ where $A_k = \int_{L_k} w_n(x-p_{L_k}) d p_{L_k}, a_k = A_k S_k(x)$, with weight function $w_n(x-p_{L_k}) = \frac{1}{\|x-p_{L_k}\|^{2n}}$ for an arbitrary $n \in \mathbb{R}$, and normal function $S_k(x) = (x-p_{L_k})^T n_{L_k}$. For any $x$:

- if $x$ is in the interior of $L$, then $f(x) < 0$;
- if $x$ is in the exterior of $L$,
  - if $n = 1$, $f(x) = 0$,
  - if $n > 1$, $f(x) > 0$,
  - if $n < 1$, $f(x) < 0$.
- if $x$ is on $L$, $f(x) = 0$.

**Proof:** This corollary follows from Theorem 3.3 and by noting that when $x$ is on $L_k \in L$, then the weight function for $L_k$ dominates and $f(x) = 0$. 

\[ \Box \]
\( n = 1/2 \) in Eq. 6  
\( n = 1 \) in Eq. 6  
\( n = 3/2 \) in Eq. 6

Fig. 6. Example of different exterior results with three weight functions. \( x \) is the query point, \( r_1 = 1 \), \( r_2 = 2 \), \( r_3 = 3 \), and \( r_4 = 4 \). Intersections are grouped into blue and orange.

An example illustrating this corollary is shown in Fig. 6.

**Corollary 3.7.** To interpolate using IMLS with the normal function in Eq. 7 in 2D, and have the property that the implicit function has positive value outside the polygonal curve, zero on the polygonal curve and negative value inside the polygonal curve, the weight function in Eq. 6 should have \( n > 1 \).

### 3.3 Necessary Conditions on the Weight Function for IMLS in 3D

In this section, we derive a 3D implicit moving least squares formulation that is equivalent to Eq. 8 by using 3D spherical coordinates. With the spherical coordinate formulation, we will show necessary conditions on the weight function for 3D IMLS approximation and interpolation to avoid degeneracy problems. While these conditions are also sufficient conditions for 2D IMLS interpolation, they are not sufficient conditions for approximation; thus we also give some sufficient (but not necessary) conditions on the approximation method.

#### 3.3.1 Spherical Coordinate Formulation of IMLS in 3D

In 3D, with an input polyhedral surface \( \Omega = \{ \Omega_k \} \), \( k = 1, \ldots, K \) and the weight function in Eq. 6, the implicit function in Eq. 8 is

\[
\sum_{k=1}^{K} \frac{a_k}{A_k}(x) 
\]

where \( A_k = \int_{\Omega_k} w_n(x - p_{\Omega_k})d\Omega_k \) and \( a_k = A_k S_k(x) \). The normal function is chosen with \( \phi = 0 \), i.e., \( S_k(x) = (x - p_{\Omega_k})^T n_{\Omega_k} \). Similar to the 2D case, in 3D we transform points into spherical coordinates \((r, \theta, \varphi)\) where the origin is the query point \( x \), \( \theta \) ranges from 0 to \( \pi \), and \( \varphi \) ranges from 0 to \( 2\pi \). Let \( r_k = \| r_k \| \) be the distance from the query point \( x \) to points on \( \Omega_k \), then

\[ S_k(x) = (x - p_{\Omega_k})^T n_{\Omega_k} = r_k^T n_{\Omega_k} = b_k \]
is the signed distance from \( x \) to \( \Omega_k \).

We perform a change of coordinates to use solid angles. Let the solid angle subtended by area \( A \) at point \( x \) be measured by the area \( \omega \) on the surface of unit sphere centered at \( x \). Let \( \psi \) be the angle between the normal to the surface and the line connecting the surface point and point \( x \). Treating \( A \) as an input polygon \( \Omega_k \), the solid angle \( d\omega_k \) subtended by \( d\Omega \) is

\[
d\omega_k = \frac{\cos \psi}{r_k^2} d\Omega_k \quad \Rightarrow \quad \omega_k = \int_{\Omega_k} \frac{\cos \psi}{r_k^2} d\Omega_k
\]

where \( \Omega_k \) here stands for the \( k \)th input polygon (see Long [Lon92] for details).

Since \( \cos \psi = (x - p_{\Omega_k})^T n_{\Omega_k} / \| x - p_{\Omega_k} \| = | b_k / r_k \), hence \( d\omega_k = | b_k / r_k^2 d\Omega_k \). On the other hand, since \( d\omega_k \) is an area on the unit sphere, we also have \( d\omega_k = \sin \theta \, d\theta d\varphi \). So \( b_k d\Omega_k = \text{sgn}(b_k) r_k^3 d\omega_k = \text{sgn}(b_k) r_k^3 \sin \theta d\theta d\varphi \). Thus,

\[
a_k = \int_{\Omega_k} \omega_n(x - p_{\Omega_k}) b_k d\Omega_k = \text{sgn}(b_k) \int_{\Omega_k} \omega_n(r_k) r_k^3 \sin \theta d\theta d\varphi
\]

where \( \text{sgn}(\cdot) \) is the sign function. We will use Eq. (22) in the next section to give necessary conditions on the weight function.

### 3.3.2 Necessary Conditions on the Weight Function for 3D Approximation

In this section, we will use a similar approach as in 2D to give a bound for the weight function in Eq. (6) to avoid degeneracies in IMLS approximation in 3D.

**Theorem 3.8.** In 3D, for the implicit moving least squares method with an input manifold polyhedron \( \Omega = \{ \Omega_k \}, k = 1, \ldots, K, p_{\Omega_k} \) being a point on polygon \( \Omega_k \), and \( x \) being the point of evaluation, let the implicit function be \( f(x) = \sum_{k=1}^{K} a_k / \sum_{k=1}^{K} A_k \) where \( A_k = \int_{\Omega_k} w(x - p_{\Omega_k}) d\Omega_k, a_k = A_k S_k(x) \), with weight function \( w(x - p_{\Omega_k}) = \left( \frac{1}{\| x - p_{\Omega_k} \|^2 + \epsilon^2} \right)^n \) for an arbitrary \( n \in \mathbb{R} \), and normal function \( S_k(x) = (x - p_{\Omega_k})^T n_{\Omega_k} \). Then

- when \( n < 3/2 \), or \( n = 3/2 \) and \( \epsilon \neq 0 \), for any \( x \), \( f(x) < 0 \);
- when \( n = 3/2 \) and \( \epsilon = 0 \):
  - if \( x \) is in the exterior of \( \Omega \) or lies on \( \Omega \), then \( f(x) = 0 \);
  - if \( x \) is in the interior of \( \Omega \), then \( f(x) < 0 \);
- when \( n > 3/2 \), for any \( x \), let \( r_{\min}(x) = \min_{p \in \Omega} \| x - p \| \) and \( r_{\max}(x) = \max_{p \in \Omega} \| x - p \| \):
  - if \( r_{\max}(x) \leq \sqrt{3\epsilon^2/(2n - 3)} \), then \( f(x) < 0 \);
  - if \( x \) is in the exterior of \( \Omega \) and \( r_{\min}(x) \geq \sqrt{3\epsilon^2/(2n - 3)} \), then \( f(x) > 0 \);
  - if \( x \) is in the interior of \( \Omega \), and \( r_{\min}(x) \geq \sqrt{3\epsilon^2/(2n - 3)} \) then \( f(x) < 0 \).

**Proof:** Since \( A_k = \int_{\Omega_k} \omega_n(x - p_{\Omega_k}) d\Omega_k \) is always positive, the sign of the implicit function \( f(x) \) in Eq. (3.8) is determined by the numerator \( \sum_{k=1}^{K} a_k \). According to Eq. (22)

\[
\sum_{k=1}^{K} a_k = \sum_{k=1}^{K} \text{sgn}(b_k) \int_{\Omega_k} \omega_n(r_k) r_k^3 \sin \theta d\theta d\varphi.
\]

For any \((\theta, \varphi), w_n(r_k) r_k^3 \) can be computed by casting a ray \( \tilde{r} \) from \( x \) in direction \((\theta, \varphi)\), and intersecting \( \Omega \) with \( \Omega \). Let \( \ell_i \) be the \( i \)th intersection of \( \tilde{r} \) with \( \Omega \) as we step along \( \tilde{r} \) from \( x \), where the value of \( \ell_i \) is the index of the polygon intersected by \( \tilde{r} \). The set of intersections for a ray in direction \((\theta, \varphi)\) from \( x \) is represented as \( R(\theta, \varphi) = \{ \ell_i \}, i = 1, 2, \ldots, N \), where \( N \) depends on \((\theta, \varphi)\). Therefore, the sum of the \( a_k \)’s is

\[
\sum_{k=1}^{K} a_k = \int \sin \theta \sum_{\ell \in R(\theta, \varphi)} \text{sgn}(b_k) \frac{r_{\ell}^3}{(r_{\ell}^2 + \epsilon^2)^n} d\theta d\varphi.
\]
As we step along the ray from $x$, the set $R(\theta, \varphi)$ is in ascending order in the distance from $x$, i.e., $r_{\ell_i}(\theta, \varphi) < r_{\ell_j}(\theta, \varphi)$ for $i < j$. From Eq. 24, similar to the proof in 2D, the sign of $f$ can be determined by

$$I(\theta, \varphi) = \sum_{j=1}^{N} \text{sgn}(b_{\ell_j}) \frac{r_{\ell_j}^3}{(r_{\ell_j}^2 + \epsilon^2)^n}. \quad (25)$$

By the Jordan-Brouwer Theorem, we know that space is divided by the surface into two component, the bounded interior and the unbounded exterior. There are a finite number of intersections located exactly on shared vertices or shared edges of two or more polygons. Among these intersections, if the ray does not transit between interior and exterior at this vertex, then the intersection is discarded. When the ray is in the plane of a polygon, the implicit value contribution of the polygon is always zero. Discarding these intersections in $R(\theta, \varphi)$ will not change the integral of Eq. 24. Thereafter, if $x$ is in the exterior of $\Omega$, $N$ is even, and if the origin is in the interior of $\Omega$, $N$ is odd.

When $x$ is in the exterior, $\text{sgn}(b_{\ell_j}) = (-1)^{j+1}$ where $\ell_j \in R(\theta, \varphi)$. Eq. 25 becomes

$$I(\theta, \varphi) = \sum_{j=1}^{N} (-1)^{j+1} \frac{r_{\ell_j}^3}{(r_{\ell_j}^2 + \epsilon^2)^n}, \quad (26)$$

where $N$ is even. From Lemma 3.1 with $t = 3$,

- If $n < 3/2$ or $n = 3/2$ and $\epsilon \neq 0$ then $I(\theta, \varphi) < 0$ for each $\theta$, thus $f(x) < 0$;
- If $n = 3/2$ and $\epsilon = 0$ then $I(\theta, \varphi) = 0$ for all $\theta$, $\varphi$, thus $f(x) = 0$;
- If $n > 3/2$ and $r_{\max} < \sqrt{3}\epsilon^2/(2n-3)$, then $r_{\ell_N}(\theta, \varphi) < \sqrt{3}\epsilon^2/(2n-3)$ and thus $I(\theta, \varphi) < 0$ for each $(\theta, \varphi)$. So $f(x) < 0$.
- If $n > 3/2$ and $r_{\min} > \sqrt{3}\epsilon^2/(2n-3)$, then $r_{\ell_1}(\theta, \varphi) > \sqrt{3}\epsilon^2/(2n-3)$ and thus $I(\theta, \varphi) > 0$ for each $(\theta, \varphi)$. So $f(x) > 0$.

When $x$ is in the interior of $\Omega$, $\text{sgn}(b_{\ell_j}) = (-1)^j$ where $\ell_j \in R(\theta, \varphi)$. Then,

$$I(\theta, \varphi) = \sum_{j=1}^{N} (-1)^j \frac{r_{\ell_j}^3}{(r_{\ell_j}^2 + \epsilon^2)^n}, \quad (27)$$

where $N$ is odd. From Lemma 3.2 with $t = 3$ and $\epsilon \neq 0$,

- If $n \leq 3/2$ then $I(\theta, \varphi) < 0$ for each $\theta$, thus $f(x) < 0$;
- If $n > 3/2$ and $r_{\max} \leq \sqrt{3}\epsilon^2/(2n-3)$, then $r_{\ell_N}(\theta, \varphi) \leq \sqrt{3}\epsilon^2/(2n-3)$ and $I(\theta, \varphi) < 0$ for each $(\theta, \varphi)$. So $f(x) < 0$.
- If $n > 3/2$ and $r_{\min} \geq \sqrt{3}\epsilon^2/(2n-3)$, then $r_{\ell_1}(\theta, \varphi) \geq \sqrt{3}\epsilon^2/(2n-3)$ and $I(\theta, \varphi) < 0$ for every $(\theta, \varphi)$. So $f(x) < 0$.

When $x$ is in the interior of $\Omega$, $N$ is odd and $\epsilon = 0$, from Lemma 3.2 we know $I(\theta, \varphi) < 0$, so $f(x)$ is negative.

If $x$ is on the input polyhedron then $x$ is on a particular polygon $\Omega_{k'}$. The intersection of $\vec{r}$ with $\Omega_{k'}$ has no contribution to $\sum_{k=1}^{K} a_k$ because $b_{k'} = 0$ and $a_{k'} = 0$. Therefore, for every ray cast from $x$ to $\Omega_{k'}$, the first intersection on $\Omega_{k'}$ is ignored. When the direction of the ray is towards the outside (inside) of $\Omega$, it is the same case as when $x$ is in the exterior (interior) of $\Omega$. Therefore, for $x$ on $\Omega$, the sign of $f(x)$ is determined only in those cases when the interior and exterior have the same sign, i.e., when $n \leq 3/2$, $f(x) < 0$; when $n > 3/2$ and $r_{\max} < \sqrt{3}\epsilon^2/(2n-3)$, $f(x) < 0$.

As in the 2D case, what Theorem 3.8 tells us is that for $n > 3/2$, there is an unbounded region outside of $\Omega$ where $f$ is strictly positive and when certain conditions are met, there is a bounded region inside $\Omega$ where $f$ is strictly negative. This leads to the following corollaries.
Corollary 3.9. When using IMLS to approximate a manifold polyhedron ($\epsilon \neq 0$), if $n > 3/2$ and if there exists a point $x$ inside of $\Omega$ such that $r_{\min} \geq \sqrt{3\epsilon^2 / (2n - 3)}$ or $r_{\max} \leq \sqrt{3\epsilon^2 / (2n - 3)}$, then the resulting implicit function $f$ is well-defined.

Corollary 3.10. When using IMLS to approximate a convex polyhedron ($\epsilon \neq 0$) if $n > 3/2$ then the resulting implicit function $f$ is well-defined. In addition, the interior of $\Omega$ is in the interior of $f$.

As in 2D, Theorem 3.8 tells us when $f$ is well-defined. Whether polyhedrons exist where $f$ is not well-defined when $n > 3/2$ is an open question.

Again, the case when $\epsilon = 0$ is of particular interest, since it results in interpolation.

Corollary 3.11. In 3D, for the implicit moving least squares method interpolating a manifold polyhedral surface defined by a set of polygons $\Omega = \{\Omega_k\}, k \in [1, K]$, $p_{\Omega_k}$ being a point on polygon $\Omega_k$, and $x$ being the evaluated point, let the implicit function be $f(x) = (\sum_{k=1}^{K} a_k) / (\sum_{k=1}^{K} A_k)$ where $A_k = \int_{\Omega_k} w_n(x - p_{\Omega_k}) d\Omega_{\Omega_k}$, $a_k = A_k S_k(x)$, with weight function $w_n(x - p) = \frac{1}{\|x - p\|^n}$ for $n \in \mathbb{R}$ and shape function $S_k(x) = (x - p_{\Omega_k})^T n_{\Omega_k}$. For any $x$:

- if $x$ is in the interior of $\Omega$, then $f(x) < 0$;
- if $x$ is in the exterior of $\Omega$,
  - if $n > 3/2$, $f(x) > 0$,
  - if $n = 3/2$, $f(x) = 0$,
  - if $n < 3/2$, $f(x) < 0$.
- if $x$ in on $\Omega$, $f(x) = 0$.

Proof: This corollary follows from Theorem 3.8 and by noting that when $x$ is on $\Omega_k \in \Omega$, then the weight function for $\Omega_k$ dominates and $f(x) = 0$.

3.4 Discussion

For interpolation, Corollaries 3.6 and 3.11 give necessary and sufficient conditions ($n > 1$ in 2D and $n > 3/2$ in 3D) for the IMLS method of Shen et al. to construct a well-defined implicit function for a manifold polygon or manifold polyhedron. For approximation, Theorems 3.3 and 3.8 show that these are also necessary conditions for the IMLS method to construct a well-defined implicit function for a manifold polygon or manifold polyhedron; these are also sufficient conditions for approximation if the polygon or polyhedron is convex, but it is unknown whether or not these are sufficient conditions for approximating concave data.

Corollaries 3.4 and 3.9 provide sufficient conditions for IMLS to approximate a closed polygon/polyhedron with a well-defined implicit function, but these conditions are not necessary. For example, Fig. 7 shows the result of approximating a $0.58 \times 0.58$ square using $\epsilon = 0.3$ and $n = 2$ in the weight function in Eq. 6. The threshold distance according to Corollary 3.4 is $\sqrt{\epsilon^2 / (n - 1)} = 0.3$. For any point in the interior of the square, $r_{\min} \leq 0.29$ and $r_{\max} \geq 0.29\sqrt{2}$, and thus the conditions of the corollary are not met. However, since this is a convex shape, the conditions of Corollary 3.5 are met, so $f$ is well-defined.

In addition, the above results explain the problems that occurred in Fig. 1 using the method of Park et al. [PLK12]. Park et al. chose $n = 1$ for 2D IMLS interpolation, and according to Corollary 3.6, when $n = 1$, the implicit values are 0 everywhere outside the polygon. From Theorem 3.3, when $n = 1$, for 2D approximation, the implicit function is negative everywhere. Further, Shen et al. [SOS04] used $n = 2$ without providing a reason in their paper. Theorem 3.8 and Corollary 3.11 show that $n = 2$ is necessary to achieve a well-defined implicit function (i.e., they could not have used $n = 1$).
Fig. 7. An example where the conditions in Corollary 3.4 are not met but a well-defined approximation curve is generated. Approximating with $\epsilon = 0.3$ and $n = 2$. The solid curve is the approximation and dashed square is the input data. The size of the square is $0.58 \times 0.58$.

4 Conclusion and Future Work

We have presented a theory about the weight function in the implicit moving least squares method, which is helpful to the understanding of IMLS based surface reconstruction approaches. Our result explains the degeneracy that occurs in Park et al. [PLK12] and why Shen et al. [SOS04] chose $n = 2$. Although we proved only 2D and 3D cases in this paper, our theorems can be generalized to higher dimensional cases in the same way.

Our proof for choosing weight functions of IMLS requires the polygonal curve and polyhedral surface to be manifold. When polygon soup contains holes and self-intersections, our proof is not applicable. Another avenue for future research is to extend our proofs to polygon soup. This will likely be non-trivial since polygon soup is usually not closed, and thus it is hard to distinguish the inside from the outside. Regardless, we expect that our conditions on manifold polyhedron are necessary conditions on polygon soup, but that additional conditions will be required to guarantee that the implicit function constructed for polygon soup is well-defined.

For $n = 1$ and $\epsilon = 0$, Park et al.’s method builds an implicit function that is 0 at every point in the exterior of the polygon, shown in Fig. 1. While this makes the method unsuitable for data interpolation and approximation, note that this is a function that has compact support. Potentially, such a function is useful for other applications.

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Appendix

Proof of Lemma 3.1:

Proof: Each term in Eq. 9 is of the form $h(r) = \frac{r^t}{(r^2 + \epsilon^2)^n}$, so Eq. 9 can be transformed to $I = \sum_{i=1}^{N/2} h(r_{2i-1}) - h(r_{2i})$. The derivative of function $h(r)$ is

$$h'(r) = \frac{tr^{t-1}(r^2 + \epsilon^2)^n - 2ntr^{t+1}(r^2 + \epsilon^2)^{n-1}}{(r^2 + \epsilon^2)^{2n}} = \left((t-2n)r^2 + t\epsilon^2\right)\frac{r^{t-1}}{(r^2 + \epsilon^2)^{n+1}}.$$ (28)
If \( h'(r) > 0 \), then \( h(r) \) is monotonically increasing and \( h(2i-1) - h(2i) < 0, \forall i \). So, \( I < 0 \).
Similarly, if \( h'(r) < 0 \), then \( I > 0 \) and if \( h'(r) = 0 \) then \( I = 0 \). Because the sign of \( h'(r) \) is depend only on the sign of \((t-2n)r^2 + tε^2\), each statement in this lemma can be verified.

Proof of Lemma 3.2:
Proof: Let \( h(r) = \frac{r^t}{(r^2 + ε^2)^{t/r}} \), then Eq. 10 can be transformed to two equivalent formulations:
\[ I = -h(r_1) + I_1 \text{ where } I_1 = \sum_{i=1}^{(N-1)/2} h(r_{2i}) - h(r_{2i+1}), \text{ and } I = -h(r_N) - I_2 \text{ where } I_2 = \sum_{i=1}^{(N-1)/2} h(r_{2i-1}) - h(r_{2i}). \]
Note that both \( I_1 \) and \( I_2 \) are the summation of \( N-1 \) (even) elements. When \( ε = 0 \), by Lemma 3.1 we have
- if \( n < t/2 \), \( I_1 < 0 \); so \( I = -h(r_1) + I_1 < 0 \);
- if \( n = t/2 \), \( I_1 = 0 \); so \( I = -h(r_1) < 0 \);
- if \( n > t/2 \), \( I_2 > 0 \); so \( I = -h(r_N) - I_2 < 0 \).
Therefore, \( I < 0 \) when \( ε = 0 \). When \( ε ≠ 0 \), similarly according to Lemma 3.1,
- if \( n < t/2 \), \( I_1 < 0 \); so \( I = -h(r_1) + I_1 < 0 \);
- if \( n > t/2 \) and \( r_N ≤ \sqrt{tε^2/(2n-t)} \), \( I_1 < 0 \); so \( I = -h(r_1) + I_1 < 0 \);
- if \( n > t/2 \) and \( r_1 ≥ \sqrt{tε^2/(2n-t)} \), \( I_2 > 0 \); so \( I = -h(r_N) - I_2 < 0 \).

References