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Griffiths Groups of Supersingular Abelian Varieties

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Abstract. The Griffiths group $\mathrm{Gr}^r(X)$ of a smooth projective variety X over an algebraically closed field is defined to be the group of homologically trivial algebraic cycles of codimension r on X modulo the subgroup of algebraically trivial algebraic cycles. The main result of this paper is that the Griffiths group $\mathrm{Gr}^2(A_{\bar{k}})$ of a supersingular abelian variety $A_{\bar{k}}$ over the algebraic closure of a finite field of characteristic p is at most a p-primary torsion group. As a corollary the same conclusion holds for supersingular Fermat threefolds. In contrast, using methods of C. Schoen it is also shown that if the Tate conjecture is valid for all smooth projective surfaces and all finite extensions of the finite ground field k of characteristic $p > 2$, then the Griffiths group of any ordinary abelian threefold $A_{\bar{k}}$ over the algebraic closure of k is non-trivial; in fact, for all but a finite number of primes $\ell \neq p$ it is the case that $\mathrm{Gr}^2(A_{\bar{k}}) \otimes \mathbb{Z}_\ell \neq 0$.

1 Introduction

Let k be a finite field of characteristic $p > 0$. We fix an algebraic closure $\bar{k}$ of k. For any finite extension k'/k we write $G_{k'}$ for the Galois group of $\bar{k}/k'$. Let X/k be a smooth, projective variety over k. We will write $X_{\bar{k}}$ for $X \times_k \bar{k}$. Let $Z^r(X_{\bar{k}})$ be the group of codimension $r \geq 0$ cycles on $X_{\bar{k}}$. Let $Z^r(X_{\bar{k}})_{\mathrm{rat}}$, $Z^r(X_{\bar{k}})_{\mathrm{alg}}$ and $Z^r(X_{\bar{k}})_{\mathrm{hom}}$ be the subgroups of codimension r cycles which are rationally, respectively algebraically, respectively homologically equivalent to zero on $X_{\bar{k}}$. We will write $\mathrm{CH}^r(X_{\bar{k}})_{\mathrm{alg}} \subset \mathrm{CH}^r(X_{\bar{k}})_{\mathrm{hom}} \subset \mathrm{CH}^r(X_{\bar{k}})$ for the corresponding groups modulo the subgroup of cycles rationally equivalent to zero. Then the quotient $\mathrm{CH}^r(X_{\bar{k}})/\,\mathrm{CH}^r(X_{\bar{k}})_{\mathrm{hom}}$ is finitely generated modulo torsion, and the Tate conjecture predicts the rank of this group to be the order of vanishing of a suitable L-function [23].

The quotient $\mathrm{CH}^r(X_{\bar{k}})_{\mathrm{hom}}/\,\mathrm{CH}^r(X_{\bar{k}})_{\mathrm{alg}}$ was first investigated by P. Griffiths and is called the *Griffiths group* of $X_{\bar{k}}$. In every example where the structure of this group is known, when it is not trivial it is quite subtle. We will write $\mathrm{Gr}^r(X_{\bar{k}})$ for the group $\mathrm{CH}^r(X_{\bar{k}})_{\mathrm{hom}}/\,\mathrm{CH}^r(X_{\bar{k}})_{\mathrm{alg}}$, and refer to it as the Griffiths group of codimension r cycles. Recently, Chad Schoen has investigated the structure of the Griffiths group of varieties over the algebraic closure of finite fields (see [19], [20]) and has shown that these groups can be infinite in several interesting situations. This note is inspired by these papers of Schoen.

Recall that over a algebraically closed field of characteristic p a *supersingular abelian variety* may be characterized by being isogenous to a product of supersingular elliptic curves (see [16, Theorem 4.2]), where an elliptic curve is said to be *supersingular* if it possesses no geometric points of order exactly p. The purpose of this

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note is to prove that the Griffiths group of any supersingular abelian variety is at most a p-primary torsion group (see Theorem 5.1). In [19, Theorem 14.4, page 45], Schoen had proved this assertion for the triple product of the Fermat cubic and $p \equiv 2 \pmod 3$. Our result also applies to all supersingular Fermat threefolds (see Theorem 5.3). One of the key ingredients in our proof is the work of N. Fakhruddin (see [9]). We hope to study the p-primary torsion in a forthcoming work.

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2 Beauville's Conjecture

Suppose A is an abelian variety over k. Then by results of Mukai, Beauville, Deninger-Murre (see [15], [1] and [6, Theorem 2.19, page 214]), the Chow groups of A when tensored with $\mathbb{Q}$ admit a finite decomposition:

$$\mathrm{CH}^i(A_{\bar{k}}) \otimes \mathbb{Q} = \bigoplus_j \mathrm{CH}^i_j(A_{\bar{k}}), \tag{2.1}$$

where $\mathrm{CH}^i_j(A_{\bar{k}})$ is the subset of $\mathrm{CH}^i(A_{\bar{k}}) \otimes \mathbb{Q}$ on which the flat pull-back of multiplication in $A_{\bar{k}}$ by an integer m acts as multiplication by m^{2i-j}.

Remark 2.1 To obtain the decomposition (2.1), it is not necessary to admit $\mathbb{Q}$-coefficients, but it suffices to invert integers which arise as denominators in the Riemann-Roch Theorem. In particular such a decomposition is available over a suitable localization of $\mathbb{Z}$. From now on we will work with this refined decomposition over a suitable localisation of $\mathbb{Z}$.

Using the work of Soulé (see [21]), Künnemann (see [13, Theorem 7.1, page 99]) proves that, except possibly for $j = 0$, all the remaining components of (2.1) are torsion.

Theorem 2.2 *Suppose A is an abelian variety over a finite field k. Then for all i* ≥ 0*:*

$$\mathrm{CH}^i(A_{\bar{k}}) \otimes \mathbb{Q} = \mathrm{CH}^i_0(A_{\bar{k}}) \tag{2.2}$$

This result is essentially a consequence of the fact that the motive of an abelian variety is pure in the sense of [21, Definition 3.1.1, page 331]. In particular, all the groups $\mathrm{CH}^i_j(A_{\bar{k}})$ except possibly $\mathrm{CH}^i_0(A_{\bar{k}})$ must be torsion.

Another version of this result, valid over any algebraically closed field (of positive characteristic), was proved by N. Fakhruddin (see [9]).

Theorem 2.3 *When X is a supersingular abelian variety over an algebraically closed field of characteristic p, then* $\mathrm{CH}^i_j(X) = 0$ *for* $j \neq 0, 1$.

3 A Result of Fakhruddin

We will also need the following result of N. Fakhruddin (see [9]).

Theorem 3.1 *Suppose A is a supersingular abelian variety over an algebraically closed field of characteristic p. Then the restriction of the cycle class map to* $\mathrm{CH}^d_0(A)$ *induces an isomorphism:*

$$\mathrm{CH}^d_0(A) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^{2d}_{\mathrm{et}}\big(A, \mathbb{Q}_\ell(d)\big)\,.$$

Remark 3.2 In particular, $\mathrm{CH}^i_1(X)$ is the kernel of the cycle class map when X is a supersingular abelian variety over any algebraically closed field.

The idea behind the proof of this theorem is that the subgroup $B^d(A)$ of $\mathrm{CH}^d_0(A)$ generated by classes of abelian subvarieties of A of codimension d on the one hand coincides with $\mathrm{CH}^d_0(A)$ and on the other hand, after tensoring with $\mathbb{Q}_\ell$ maps isomorphically onto $H^{2d}_{\mathrm{et}}\big(A, \mathbb{Q}_\ell(d)\big)$. See [9] for the details.

4 Abel-Jacobi Maps

In [2], Bloch constructed an Abel-Jacobi mapping

$$\lambda_i\colon\ \mathrm{CH}^i(X_{\bar{k}})_{\ell\text{-tors}} \to H^{2i-1}_{\mathrm{et}}\big(X_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)\big)$$

We will need the following results about this map.

***Theorem 4.1* (see [4, Corollary 4, page 775], [14, Section 8])** *Let X/k be any smooth, projective variety and $\ell \neq p$. Then the map λ_2 is injective.*

***Theorem 4.2* (see [22, Théorème 4.7.1, page 87])** *Let A/k be a supersingular abelian variety, and $\ell \neq p$. Then the restriction λ'_i of λ_i to* $\mathrm{CH}^i(A_{\bar{k}})_{\mathrm{alg},\ell\text{-tors}}$ *is surjective, and bijective if $i = 1, 2$ or* $\dim A$.

5 Supersingular Abelian Varieties

We are now in a position to prove the main theorem of this note.

Theorem 5.1 *Let $A_{\bar{k}}$ be a supersingular abelian variety over the algebraic closure of a finite field. Then* $\mathrm{Gr}^2(A_{\bar{k}})$ *is at most a p-primary torsion group.*

Proof Under the hypotheses of this theorem, the results in Section 2 together with Theorem 3.1 imply that

$$\mathrm{CH}^i_1(A_{\bar{k}}) = \mathrm{CH}^i(A_{\bar{k}})_{\mathrm{hom}}$$

is a torsion group. Thus $\mathrm{CH}^i(A_{\bar{k}})_{\mathrm{alg}}$ and $\mathrm{Gr}^i(A_{\bar{k}})$ are also torsion groups. So to prove $\mathrm{Gr}^2(A_{\bar{k}})$ has no ℓ-primary torsion for $\ell \neq p$ it will suffice to prove that $\mathrm{Gr}^2(A_{\bar{k}}) \otimes \mathbb{Z}_\ell = 0$, or, equivalently, that

$$\mathrm{CH}^2(A_{\bar{k}})_{\mathrm{hom}} \otimes \mathbb{Z}_\ell = \mathrm{CH}^2(A_{\bar{k}})_{\mathrm{alg}} \otimes \mathbb{Z}_\ell. \tag{5.1}$$

We use the Bloch-Abel-Jacobi mapping (Section 4) to prove (5.1). Consider the commutative diagram:

$$
\begin{array}{ccc}
\mathrm{CH}^2(A_{\bar{k}})_{\ell\text{-tors}} & \xrightarrow{\lambda_2} & H^3_{\mathrm{et}}\left(A_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)\right) \\
\uparrow & & \uparrow \lambda_2' \\
\mathrm{CH}^2(A_{\bar{k}})_{\mathrm{alg},\ell\text{-tors}} & = & \mathrm{CH}^2(A_{\bar{k}})_{\mathrm{alg},\ell\text{-tors}}
\end{array}
\tag{5.2}
$$

where the first vertical arrow is the natural inclusion. Then Theorem 4.1 implies that λ_2 is injective on ℓ-torsion, from which it follows that λ_2' must also be injective. On the other hand, Theorem 4.2 tells us that λ_2' is surjective. It follows that λ_2', and indeed, all the arrows in diagram (5.2), must be isomorphisms. Thus the two groups in (5.1) are equal, and the Griffiths group $\mathrm{Gr}^2(A_{\bar{k}})$ has no ℓ-primary torsion for any $\ell \neq p$. This completes the proof. ■

Corollary 5.2 *We have in the notation of Theorem 5.1:*

$$
H^3_{\mathrm{et}}\left(A_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)\right) \cong \mathrm{CH}^2_{\mathrm{hom}}(A_{\bar{k}}) \otimes \mathbb{Z}_\ell \cong N^1 H^3_{\mathrm{et}}\left(A_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)\right), \tag{5.3}
$$

where N^1 denotes the first step of the coniveau filtration.

Proof The first isomorphism follows from the proof of 5.1, while the second can be found in [14, Section 18] (also see [5]). ■

The method of proof of Theorem 5.1 also proves the corresponding result for supersingular Fermat threefolds over a finite field k. Recall that a smooth, projective Fermat threefold $X \subset \mathbb{P}^4$ is said to be *supersingular* if $H^3_{\mathrm{cris}}\left(X/W(k)\right)$ is of slope $3/2$ (see [22, Théorème 4.8.1, page 88], [8]).

Theorem 5.3 *Let k be a finite field of characteristic p and let $\bar{k}$ be an algebraic closure of k. Let X be a supersingular Fermat threefold over k of degree m. Then the Griffiths group of codimension two cycles on $X_{\bar{k}}$ is at most p-primary torsion.*

Proof The diagram (5.2) is also valid for a smooth Fermat threefold. As we are over a finite ground field the result of [21, Théorème 3] applies and so $\mathrm{CH}^2(X_{\bar{k}})_{\mathrm{hom}}$ is torsion and by [22, Théorème 4.8.1, page 88] the map

$$
\lambda_2 \colon \mathrm{CH}^2(X_{\bar{k}}) \to H^3_{\mathrm{et}}\left(X_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)\right)
$$

is surjective as X is supersingular. Then we are done by 4.1. ■

6 Ordinary Abelian Threefolds

In order to provide a contrast to the results in the supersingular case, in this section we observe that the behaviour of the Griffiths group is controlled by the slope filtration. This idea goes back to Bloch (see [3, Lecture 6, page 6.12]). Our remarks, which are no doubt well-known to experts, are inspired by the work of Schoen (see [20]). Recall that an abelian variety over a field k is said to be *ordinary* if its Hodge and Newton polygons (for the first crystalline cohomology) coincide. Equivalently an abelian variety A is ordinary if and only if the group of geometric points of order p has order $p^{\dim(A)}$.

Recall that for any smooth, projective variety X over k there are Abel-Jacobi maps (see [10], [19])

$$\alpha^r\colon \ \mathrm{CH}^r(X_{\bar{k}}) \to J^r_\ell(X), \tag{6.1}$$

where

$$J^r_\ell(X) := \lim_{k'/k} H^1\Big(G_{k'}, H^{2r-1}\big(X_{\bar{k}}, \mathbb{Z}_\ell(r)\big) \,/\, \mathrm{Tors}\Big),$$

the limit is taken over finite Galois extensions k' of k, and the cohomology is the continuous Galois cohomology (*i.e.* cocycles are continuous with respect to the topology on the Galois group and the ℓ-adic topology on the Galois modules). When k is a finite field $J^r_\ell(X)$ is a torsion group (see [19, Lemma 1.4, page 4]).

Before we begin, we remind the reader of the following variant of Bloch's result (see [3, Lecture 1]). This result is implicit in [3, Lecture 1]—we give a proof here for completeness (as we don't know any explicit reference) as it indicates the relation between the slope filtration and the behaviour of the Chow groups.

Theorem 6.1 *Let X/k be a smooth, projective surface over an uncountable algebraically closed field of characteristic p. Further assume that $H^2_{\mathrm{cris}}\big(X/W(k)\big)$ has a non-trivial slope zero part. Then $CH^2(X_k)$ is not representable.*

Proof By [3, Lecture 1], it suffices to prove that the hypothesis imply that the group of transcendental cycles in étale cohomology is non-trivial. Assume, if possible that it is trivial, that is, the group $H^2_{\mathrm{et}}\big(X, \mathbb{Q}_\ell(1)\big) \,/\, \mathrm{image}\big(NS(X_k)\big)_{\mathbb{Q}_\ell} = 0$. This says that the cycle class map is surjective. We may assume that X and a basis for $NS(X_k)$ are defined over a finitely generated subfield of k. Then by further spreading out to a finitely generated ring as our base. We can, by shrinking the base if necessary, assume that all the fibres are smooth. Then see that there is an non-empty zariski open set on the base where the Newton polygon of the second crystalline cohomology of every special fibre coincides with the Newton polygon of the generic fibre and hence has a non-trivial slope zero part (this follows from a theorem of Katz and Grothendieck [12]). For any such special fibre, which is defined over a finite field. By [11] we know that over a finite field the characteristic polynomial of frobenius on the ℓ-adic cohomology coincides with the characteristic polynomial of Frobenius on crystalline cohomology. Hence the cycle class map from the Neron-Severi group to the second crystalline cohomology cannot be surjective as the crystalline cohomology has a non-trivial slope zero part (as the image of Neron-Severi group) is contained in the slope

1 part of the crystalline cohomology). Thus one has a contradiction as the rank of Neron-Severi does not decrease under specialization. ■

Proposition 6.2 *Let k be a finite field of characteristic* $p > 2$. *Assume that the Tate conjecture is valid for all smooth projective surfaces and for all finite extensions of k. Then the Griffiths group of any smooth projective, ordinary abelian threefold A over* $\bar{k}$ *is non-trivial. More precisely, for all but finite number of primes* $\ell \neq p$, $\mathrm{Gr}^2(A_{\bar{k}}) \otimes \mathbb{Z}_\ell \neq 0$.

Proof Suppose A/k is an ordinary abelian threefold. From the work of Soulé (see [21, Théorème 3]) we know that the Chow group of homologically trivial cycles on an abelian threefold is torsion. Thus the Griffiths group is torsion as well. Then by [22, Corollaire 3.4, page 83], as $A_{\bar{k}}$ is ordinary the map

$$\lambda_2 \colon \mathrm{CH}^2(A_{\bar{k}})_{\mathrm{alg},\ell\text{-tors}} \to H^3_{\mathrm{et}}\left(A_{\bar{k}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)\right) \tag{6.2}$$

cannot be surjective because it has a non-trivial slope zero part in $H^3_{\mathrm{crys}}\left(A_{\bar{k}}/W(\bar{k})\right)$.

On the other hand we know from the work of Schoen (see [20, Theorem 0.1, page 795]) that the Abel-Jacobi map α^2 is surjective, for all but finite number of primes $\ell \neq p$, under the assumption that the Tate conjecture holds for all smooth projective surfaces. Thus it suffices to verify that the maps λ_2 and α^2 coincide on homologically trivial ℓ-power torsion cycles, which in turn follows from the construction of the map λ_2 given by Raskind (see [18, Section 2]). ■

Remark 6.3 We would like to complement the above proposition with the following example which illustrates that the p-torsion in the Griffiths group may be zero even when the abelian variety is an ordinary abelian variety over a finite field k of characteristic $p > 0$. Let E/k be an ordinary elliptic curve. It is standard result of Deuring that E admits a lifting to an elliptic curve C with complex multiplication defined over a number field (for a modern proof see [17, page 192]). Let $A = E \times_k E \times_k E$. Then A is an ordinary abelian variety. The entire discussion in [7, Remark 4.3, page 601] goes through for A, and one has that the p-adic Abel-Jacobi mapping constructed

$$\mathrm{CH}^2(A_{\bar{k}})_{\mathrm{alg}} \otimes \mathbb{Z}_p \to H^3_{\log}\left(A, \mathbb{Q}_p/\mathbb{Z}_p(2)\right) \tag{6.3}$$

where the target is the logarithmic cohomology (see [7] for the notation and terminology) is surjective. By [21, Théorème 3] we know that the the kernel of the crystalline cycle class map is torsion as X is an abelian threefold over a finite field. Hence we can apply the argument given above to deduce that the $\mathrm{Gr}^2(A_k) \otimes \mathbb{Z}_p = 0$. Thus p-torsion homologically trivial cycles may fail to carry a non-trivial filtration even in presence of non-trivial slope filtration.

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