# Performance Proof for Slotted Aloha

T. Charles Clancy
Electrical Engineering
University of Maryland
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We consider $n$ users sharing a slotted channel, with probability $p$ of transmitting in each time slot. The first goal is to determine the optimal probability $p$.

The probability of success $P_S$ for a single time slot is the probability that only one of $n$ users transmits.

$$\begin{aligned} P_S &= \tbinom{n}{1} p(1-p)^{n-1} \\ &= np(1-p)^{n-1} \end{aligned} \tag{1}$$

This equation is due to the fact that there are $\binom{n}{1}$ different users that can transmit, and the probability of that user transmitting is $p$ while the probability of the other $n-1$ users not transmitting is $(1-p)^{n-1}$.

Our goal is to find a value of $p$ that maximizes $P_S$. To accomplish this we take the derivative and solve it equal to zero.

$$\frac{d}{dp} P_S = -np(1-n)(1-p)^{n-2} + n(1-p)^{n-1} \tag{2}$$

And then solving $P_S' = 0$

$$\begin{aligned} np(n-1)(1-p)^{n-2} &= n(1-p)^{n-1} \\ p(n-1) &= 1-p \\ np - p &= 1-p \\ np &= 1 \\ p &= 1/n \end{aligned} \tag{3}$$

Substituting this back into our original equation we get the optimal $P_S$:

$$\begin{aligned} P_S &= np(1-p)^{n-1} \\ &= \left(1 - \frac{1}{n}\right)^{n-1} \end{aligned} \tag{4}$$

This gives us the probability of a successful transmission in any time slot. If we multiply $P_S$ by the number of bits per slot divided by the time per slot, we can compute the theoretical maximal rate in bits per second, which is the network capacity. The per-user capacity is this value divided by $n$.

If we are interested in the asymptotic case, we can consider the limit as $n \to \infty$. First, let's manipulate the expression a bit:

$$\begin{aligned} P_S &= \left(1 - \frac{1}{n}\right)^{n-1} \\ &= \frac{(n-1)^{n-1}}{n^{n-1}} \\ &= n^{-(n-1)} \sum_{k=0}^{n-1} \tbinom{n-1}{k} n^{n-1-k} (-1)^k \\ &= \sum_{k=0}^{n-1} \tbinom{n-1}{k} n^{-k} (-1)^k \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \cdot \frac{(n-1)!}{n^k (n-1-k)!} \end{aligned} \tag{5}$$

If we take the limit as $n \to \infty$, we can see that

$$\lim_{n \to \infty} \frac{(n-1)!}{n^k (n-1-k)!} = 1 \tag{6}$$

because the numerator's largest degree term is $n^{n-1}$ and the denominator's largest degree term is $n^k \cdot n^{n-1-k} = n^{n-1}$. Since both have coefficient 1, the limit is 1. Thus

$$\begin{aligned} \lim_{n \to \infty} P_S &= \lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \\ &= \mathbf{e}^{-1} \end{aligned} \tag{7}$$

The last step is due to the definition of the Taylor series of $\mathbf{e}^x$. Thus we have proven a rate of 0.368.