# Queueing Theory Lecture Notes

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## 1 Little's Theorem

$N(t)$ = Number of customers in queue at time $t$
$\alpha(t)$ = Number of customers who arrived in $[0,t]$
$T_i$ = Time spent in system by $i$th customer

Let

$$N_t = \frac{1}{t}\int_0^t N(\tau)d\tau \tag{1}$$

be the time average of $N(t)$. Let

$$N = \lim_{t\to\infty} N_t \tag{2}$$

be the steady-state time average. Let

$$\lambda_t = \frac{\alpha(t)}{t} \tag{3}$$

be the time average arrival rate over $[0,t]$. Let

$$\lambda = \lim_{t\to\infty} \lambda_t \tag{4}$$

be the steady-state arrival rate. Let

$$T_t = \frac{1}{\alpha(t)}\sum_{i=0}^{\alpha(t)} T_i \tag{5}$$

be the time average customer delay over $[0,t]$. Let

$$T = \lim_{t\to\infty} T_t \tag{6}$$

be the steady-state delay. Little's theorem:

$$N = \lambda T \tag{7}$$

## 2 Probabalistic Version

Define random variable

$$p_n(t) = Pr[n \text{ customers in queue at time } t] \tag{8}$$

Then let

$$\bar{N}(t) = E[p_n(t)] = \sum_{n=0}^{\infty} np_n(t) \tag{9}$$

In steady state, $p_n = \lim_{t\to\infty} p_n(t)$, $\forall n$, then

$$\bar{N} = E[p_n] = \sum_{n=0}^{\infty} np_n \tag{10}$$

Note that this is also

$$\bar{N} = \lim_{t\to\infty} \bar{N}(t) \tag{11}$$

For delay, assume knowledge of $\bar{T}_k$, average delay for customer $k$. Then

$$\bar{T} = \lim_{k\to\infty} \bar{T}_k \tag{12}$$

In *ergodic systems*, the time average equals the stead-state average, thus $N = \bar{N}$ and $T = \bar{T}$. Lastly, define

$$\lambda = \lim_{t\to\infty} \frac{E[\text{probabalistic } \alpha(t)]}{t} \tag{13}$$

Then Little's Theorem still holds:

$$N = \lambda T \tag{14}$$

## 3 Examples

Example 1: Packets arrive at $n$ nodes in a network with rates $\lambda_1, ..., \lambda_n$. $N$ is the average number of packets inside the network. Then the average delay per packet is

$$T = \frac{N}{\sum_{i=1}^{n} \lambda_i} \tag{15}$$

Also, $N_i = \lambda_i T_i$.

Example 2: $k$ packets per second arrive with the first arrival at $t = 0$. Each requires $\alpha k$ time to transmit. Processing and propagation time equal $P$. Interarrival times constant.

$$T = \alpha k + P \tag{16}$$

$$\lambda = \frac{1}{k} \tag{17}$$

$$N = \lambda T = \alpha + \frac{P}{k} \tag{18}$$

which holds only with time averages.

## 4 Poisson Processes

Used to measure the number arrivals between time 0 and time $t$. If $A(t)$ is a Poisson Process, then:

- $A(t+\tau) \geq A(t)$ for $\tau \geq 0$
- $A(t) \in I^{+}$
- For arrival rate $\lambda$, $A(t) \sim Poi(\tau; \lambda)$, then

$$P[A(t+\tau) - A(t) = n] = e^{-\lambda t} \frac{(\lambda \tau)^n}{n!} \tag{19}$$

  and the average is

$$E[Poi(\tau; \lambda)] = \tau \lambda \tag{20}$$

- If $t_n$ is the time of the $n$th arrival, and $\tau_n = t_{n+1} - t_n$ is the interarrival time, then

$$P[\tau_n = \tau] = \lambda e^{-\lambda \tau} \tag{21}$$

  which is an exponential probability density function, with mean $E[\tau_n] = \frac{1}{\lambda}$ and variance $Var[\tau_n] = \frac{1}{\lambda^2}$.

- If $A_i(t)$ are independent Poisson processes and $A_i(t) \sim Poi(t; \lambda_i)$ then if

$$\hat{A}(t) = \sum_{i=1}^{n} A_i(t) \tag{22}$$

  then the sum process has distribution $\hat{A}(t) \sim Poi(t; \hat{\lambda})$, where

$$\hat{\lambda} = \sum_{i=1}^{n} \lambda_i \tag{23}$$

## 5 Discrete-Time Markov Chains

$M[n]$ is a Markov process at time $n$. Markov means that the value of $M[n+1]$ depends only on the value of $M[n]$, and is independent of $M[0], ..., M[n-1]$. Let

$$Pr[M[n] = j | M[n-1] = i] = q_{ij} \tag{24}$$

Let $Q$ be a matrix of values $q_{ij}$ for all $i, j$. $Q$ is the transition probability matrix. Let

$$v_i[n] = Pr[M[n] = i] \tag{25}$$

and $v[n]$ be a vector of values $v_i[n]$. Then

$$v[n+1] = v[n]Q \tag{26}$$

which can be inductively expanded to

$$v[n] = v[0]Q^n \tag{27}$$

These are called the Chapman-Kolmogrov Equations. To find the steady-state probability, compute

$$v = \lim_{n \to \infty} v[n] = \lim_{n \to \infty} v[0]Q^n \tag{28}$$

## 6 Continuous-Time Markov Chains (CTMC)

$M(t)$ is a Markov process at time $t$. The time spent in state $i$ is distributed $exp(\lambda_i)$, where $\lambda_i$ is the *exit rate* from state $i$. The probability of transitioning to state $j$ from state $i$ is, as before, $q_{ij}$. Let $\lambda$ be a vector of $\lambda_i$. Then the rate of transition between states is $\lambda Q$.

Let $p_j = P$[in state $j$ at steady-state] and $T_j(t)$ be the time in state $j$ from $[0, t]$, then

$$p_j = \lim_{t \to \infty} \frac{T_j(t)}{t} = \lim_{t \to \infty} Pr[M(t) = j] \tag{29}$$

## 7 CTMC Queues

$M(t)$ is the number of customers in the system. The rate at which $M(t)$ increases by one is $\lambda$ and the rate at which it decreases is $\mu$, which are the arrival and departure rates from the system. Our goal is to compute

$$p_j = \lim_{t \to \infty} Pr[M(t) = j] \tag{30}$$

Look at the transitions from state $n$ to state $n+1$. They must be within 1 of each other. Thus

$$p_n \lambda = p_{n+1} \mu \tag{31}$$

If we let $\rho = \lambda / \mu$, then

$$p_{n+1} = \rho p_n \tag{32}$$

Inductively, we can compute that

$$p_n = \rho^n p_0 \tag{33}$$

For $\rho < 1$ we have

$$1 = \sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \rho^n p_0 = \frac{p_0}{1-\rho} \tag{34}$$

Solving, we get $p_0 = 1 - \rho$ and $p_n = \rho^n (1 - rho)$.

Looking at the steady-state queue length

$$
\begin{aligned}
N &= \lim_{t\to\infty} E[N(t)] \\
&= \sum_{n=0}^{\infty} np_n \\
&= \sum_{n=0}^{\infty} n\rho^n(1-\rho) \\
&= \rho(1-\rho)\sum_{n=0}^{\infty} n\rho^{n-1} \\
&= \rho(1-\rho)\frac{d}{d\rho}\left(\sum_{n=0}^{\infty}\rho^n\right) \\
&= \rho(1-\rho)\frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) \\
&= \rho(1-\rho)\frac{1}{(1-\rho)^2} \\
&= \frac{\rho}{1-\rho} \\
&= \frac{\lambda}{\mu-\lambda}
\end{aligned} \tag{35}
$$

Then the system wait time

$$T = \frac{N}{\lambda} = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu-\lambda} \tag{36}$$

The average queue wait time

$$W = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\rho}{\mu-\lambda} \tag{37}$$

And the number in the queue

$$N_Q = \lambda W = \frac{\rho^2}{1-\rho} \tag{38}$$