Fourier Transform

- Analytic geometry gives a coordinate system for describing geometric objects.
- Fourier transform gives a coordinate system for functions.

Basis

- P=(x,y) means P = x(1,0)+y(0,1)
- Similarly:

$$f(\theta) = a_{11}\cos(\theta) + a_{12}\sin(\theta)$$
$$+ a_{21}\cos(2\theta) + a_{22}\sin(2\theta) + \ldots$$

Note, I'm showing non-standard basis, these are from basis using complex functions.

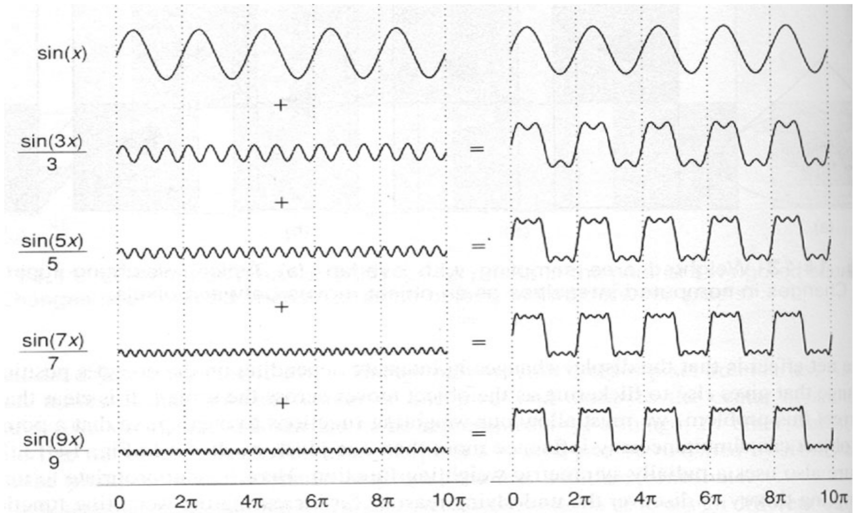


Image from *Computer Graphics: Principles and Practice*
by Foley, van Dam, Feiner, and Hughes

Orthonormal Basis

- ||(1,0)||=||(0,1)||=1
- (1,0).(0,1)=0
- Similarly we use normal basis elements eg:

$$\frac{\cos(\theta)}{\left\|\cos(\theta)\right\|} \qquad \left\|\cos(\theta)\right\| = \sqrt{\int_0^{2\pi} \cos^2\theta\, d\theta}$$

- While, eg:

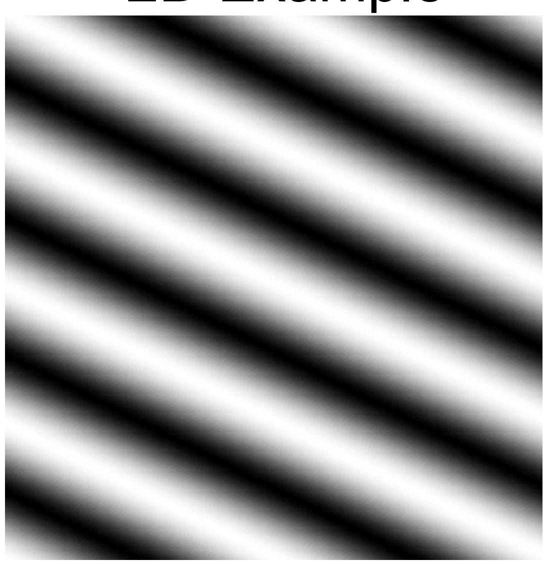
$$\int_0^{2\pi} \cos\theta \sin\theta \; d\theta = 0 \qquad \int_0^{2\pi} \cos\theta \cos 2\theta \; d\theta = 0$$

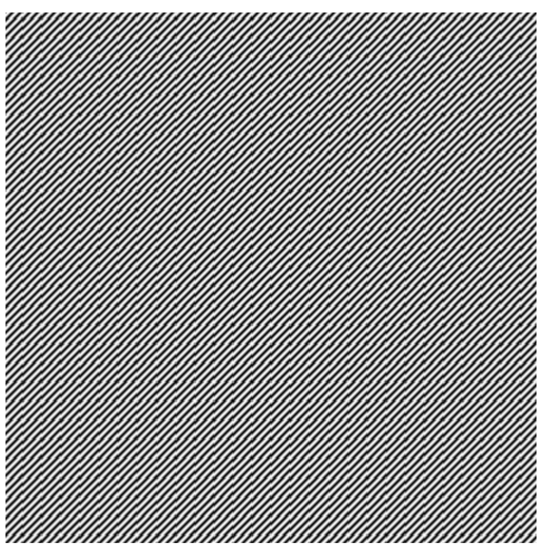
Coordinates with Inner Products

$$x = (x, y) \bullet (1,0) \quad y = (x, y) \bullet (0,1)$$
$$(x, y) = x(1,0) + y(0,1)$$

$$a_{i,1} = \int_0^{2\pi} f * \frac{\cos i\theta}{\|\cos i\theta\|} d\theta \quad a_{i,2} = \int_0^{2\pi} f * \frac{\sin i\theta}{\|\sin i\theta\|} d\theta$$
$$f = \sum a_{i,1} \frac{\cos i\theta}{\|\cos i\theta\|} + \sum a_{i,2} \frac{\sin i\theta}{\|\sin i\theta\|}$$

2D Example





Convolution Theorem

$$f \otimes g = T^{-1} F * G$$

- *F,G* are transform of *f,g*

That is, *F* contains coefficients, when we write *f* as linear combinations of harmonic basis.

This says convolution is equivalent to multiplication in the transform domain.

Examples

$$\cos\theta \otimes \cos\theta = ?$$

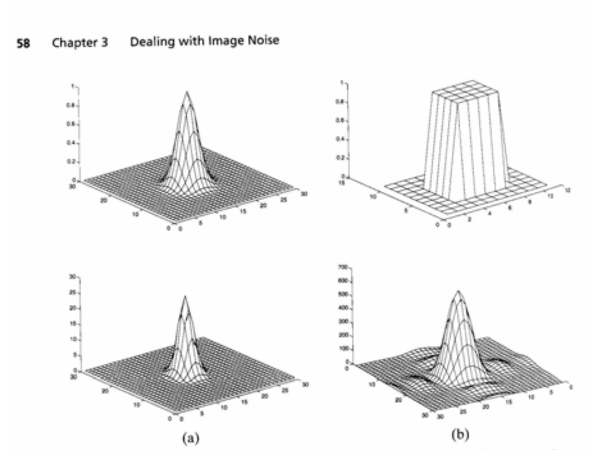
$$\cos\theta \otimes \cos 2\theta = ?$$

$$\cos\theta \otimes f = ?$$

$$(\cos\theta + .2\cos 2\theta + .1\cos 3\theta) \otimes f = ?$$

Examples

- Transform of box filter is sinc.
- Transform of Gaussian is Gaussian.

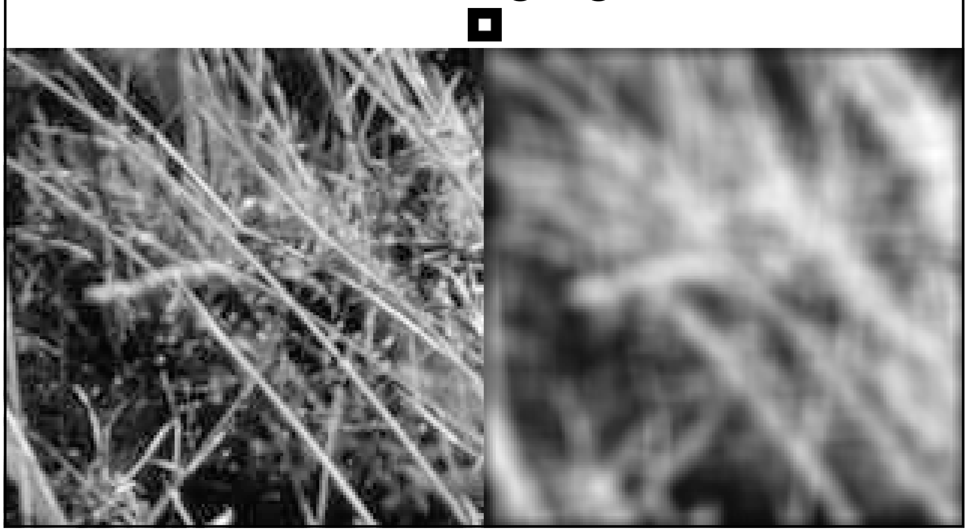


(Trucco and Verri)

Implications

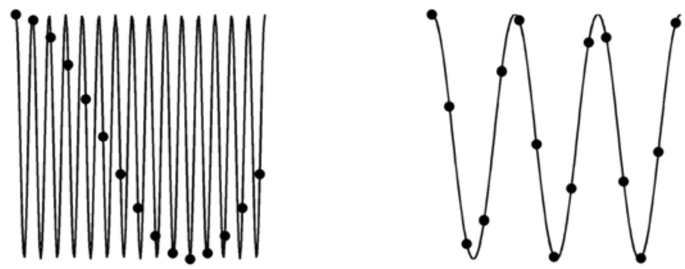
- Smoothing means removing high frequencies.
  - One definition of smooth is low-frequency.
  - This is also one definition of scale.
- Sinc function explains artifacts.
- Need smoothing before subsampling to avoid aliasing.

Example: Smoothing by Averaging



Smoothing with a Gaussian

Sampling



Every sample gives a linear equation in $a_{i,j}$.

Need two samples for every frequency.