Fourier Transform

- Analytic geometry gives a coordinate system for describing geometric objects.
- Fourier transform gives a coordinate system for functions.

Basis

- $P=(x,y)$ means $P = x(1,0) + y(0,1)$
- Similarly:

$$f(\theta) = a_{11} \cos(\theta) + a_{12} \sin(\theta)$$
$$+ a_{21} \cos(2\theta) + a_{22} \sin(2\theta) + \ldots$$

Note, I'm showing non-standard basis, these are from basis using complex functions.
Orthonormal Basis

- $\| (1,0) \| = \| (0,1) \| = 1$
- $(1,0) \cdot (0,1) = 0$
- Similarly we use normal basis elements eg:

$$\frac{\cos(\theta)}{\| \cos(\theta) \|} \quad \| \cos(\theta) \| = \sqrt{\int_0^{2\pi} \cos^2 \theta \, d\theta}$$

- While, eg:

$$\int_0^{2\pi} \cos \theta \sin \theta \, d\theta = 0 \quad \int_0^{2\pi} \cos \theta \cos 2\theta \, d\theta = 0$$
Coordinates with Inner Products

\[ x = (x, y) \cdot (1, 0) \quad y = (x, y) \cdot (0, 1) \]
\[ (x, y) = x(1, 0) + y(0, 1) \]

\[ a_{i,1} = \int_{0}^{2\pi} f * \frac{\cos i \theta}{\|\cos i \theta\|} \, d\theta \quad a_{i,2} = \int_{0}^{2\pi} f * \frac{\sin i \theta}{\|\sin i \theta\|} \, d\theta \]

\[ f = \sum a_{i,1} \frac{\cos i \theta}{\|\cos i \theta\|} + \sum a_{i,2} \frac{\sin i \theta}{\|\sin i \theta\|} \]
Convolution Theorem

\[ f \otimes g = T^{-1} F \ast G \]

- \(F, G\) are transform of \(f, g\)

That is, \(F\) contains coefficients, when we write \(f\) as linear combinations of harmonic basis.

This says convolution is equivalent to multiplication in the transform domain.

Examples

\[ \cos \theta \otimes \cos \theta = ? \]

\[ \cos \theta \otimes \cos 2\theta = ? \]

\[ \cos \theta \otimes f = ? \]

\[ (\cos \theta + .2 \cos 2\theta + .1 \cos 3\theta) \otimes f = ? \]
Examples

• Transform of box filter is sinc.
• Transform of Gaussian is Gaussian.

(Trucco and Verri)

Implications

• Smoothing means removing high frequencies.
  – One definition of smooth is low-frequency.
  – This is also one definition of scale.
• Sinc function explains artifacts.
• Need smoothing before subsampling to avoid aliasing.
Example: Smoothing by Averaging

Smoothing with a Gaussian
Sampling

Every sample gives a linear equation in $a_{i,j}$.

Need two samples for every frequency.