Linear Algebra and SVD

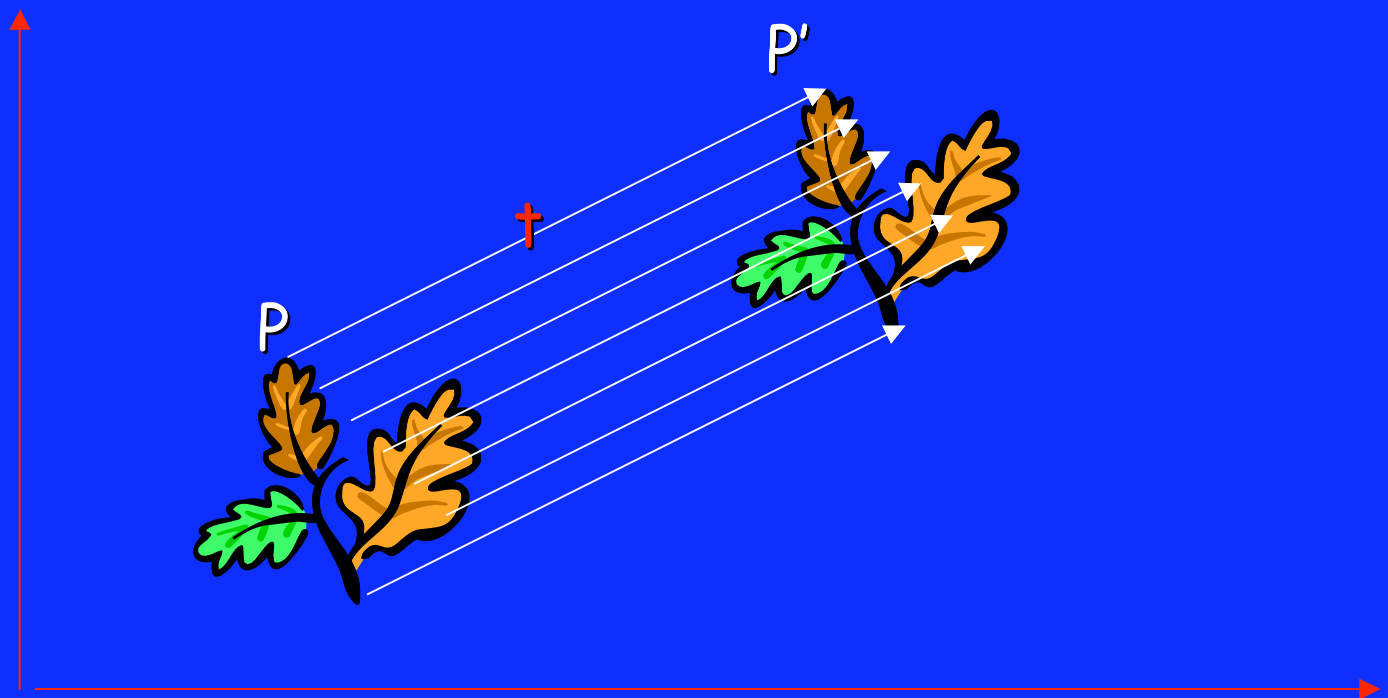
(Some slides adapted from Octavia Camps)

Goals

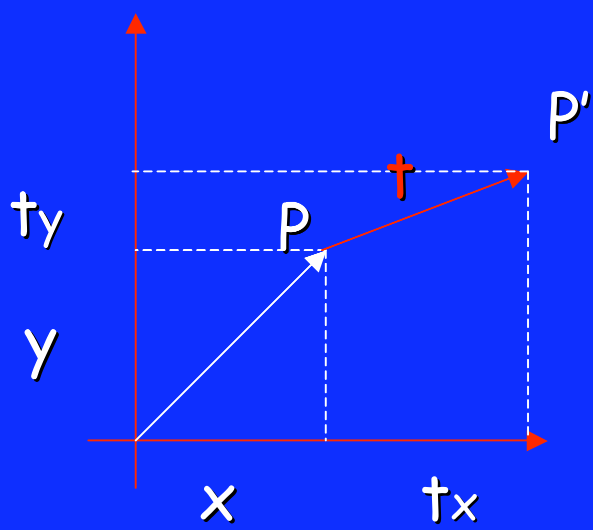
- Represent points as column vectors.
- Represent motion as matrices.
- Move geometric objects with matrix multiplication.
- Introduce SVD

Euclidean transformations

2D Translation



2D Translation Equation

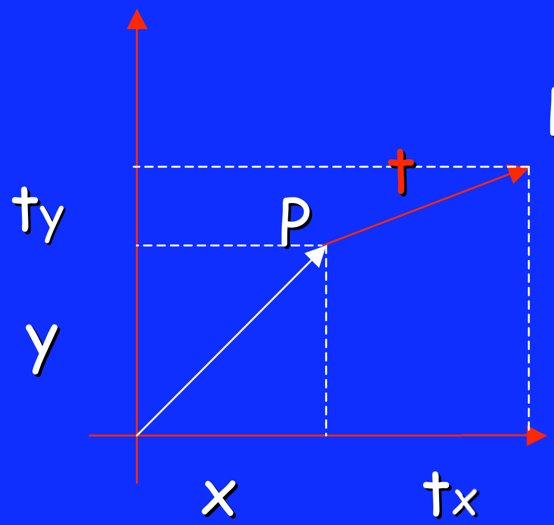


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = (x + t_x, y + t_y) = \mathbf{P} + \mathbf{t}$$

2D Translation using Matrices



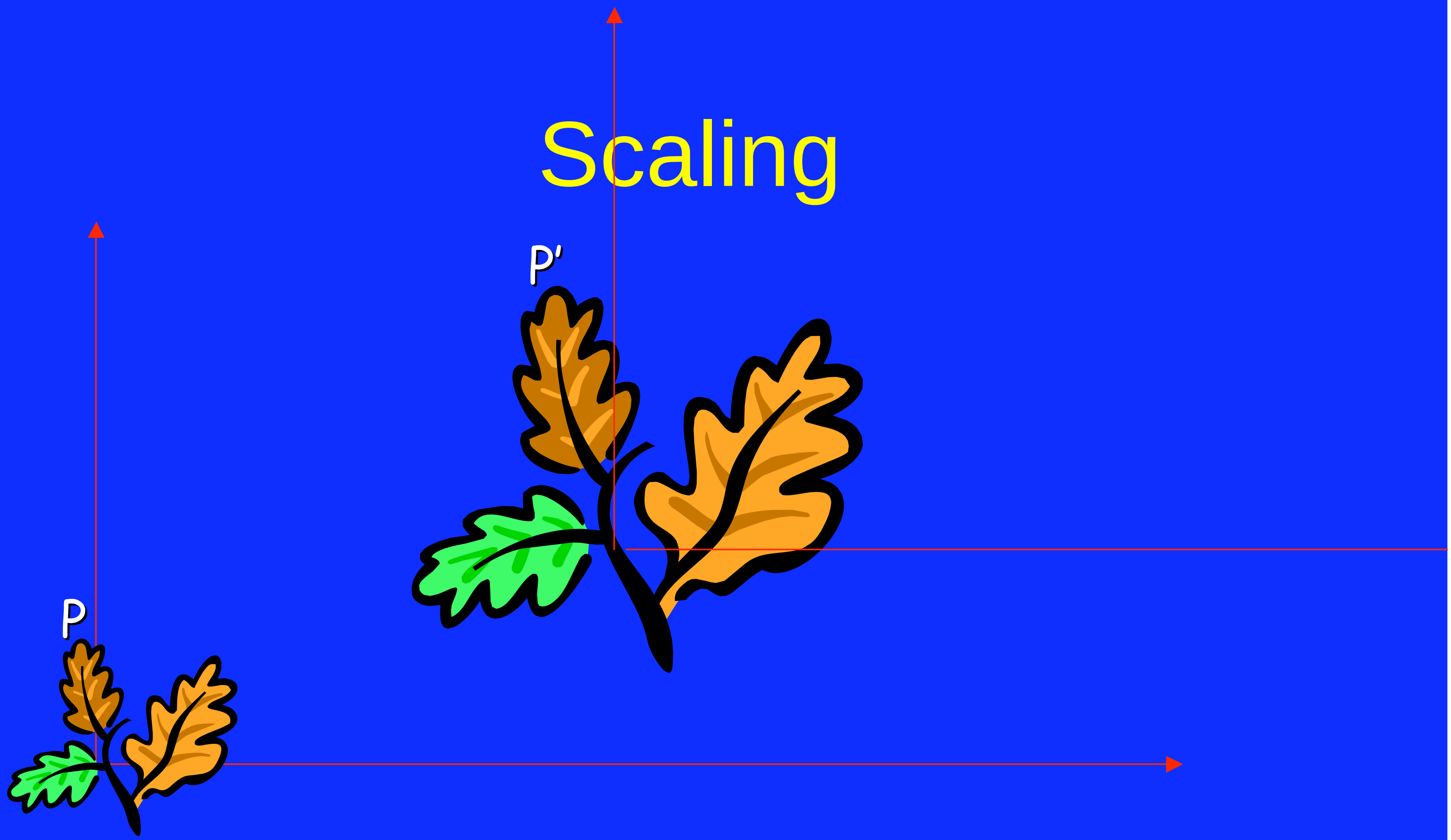
$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

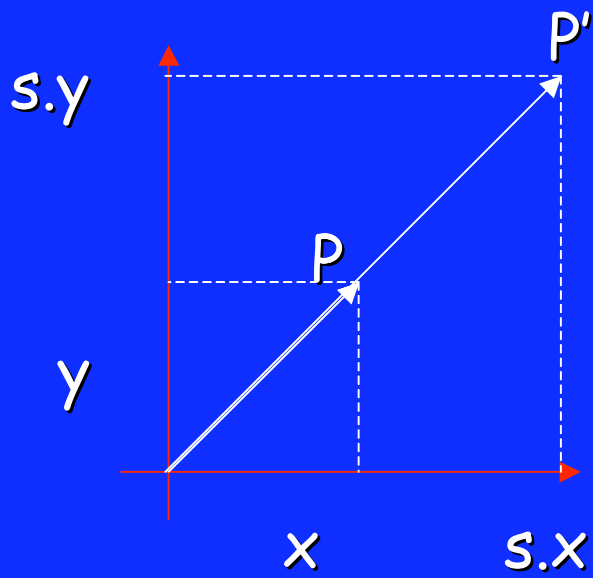
t P

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling



Scaling Equation



$$\mathbf{P} = (x, y)$$

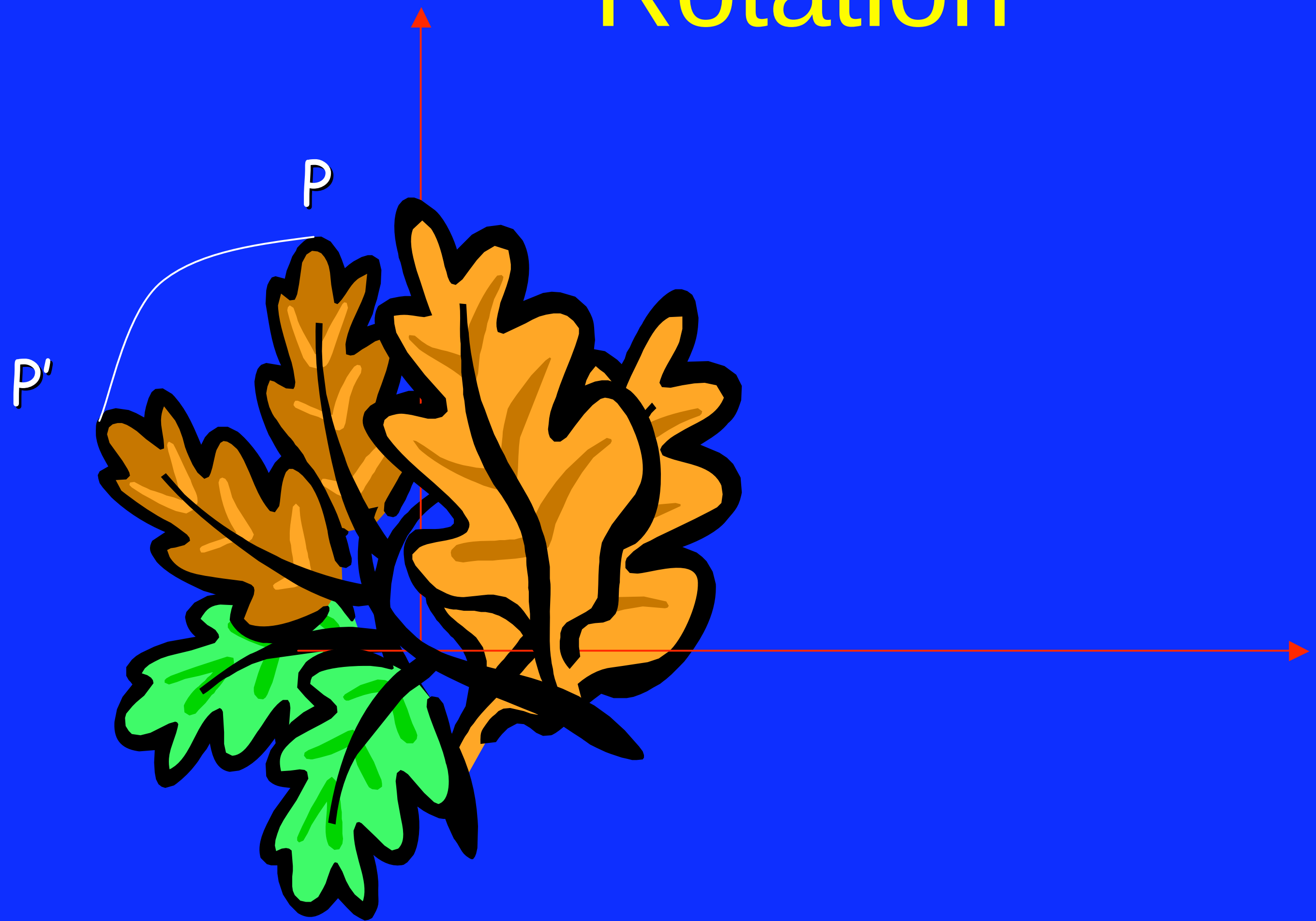
$$\mathbf{P}' = (sx, sy)$$

$$\mathbf{P}' = s \cdot \mathbf{P}$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} sx \\ sy \end{bmatrix} = \underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

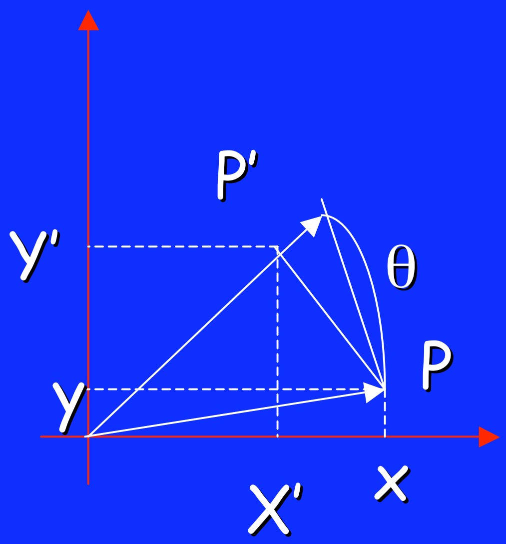
$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

Rotation



Rotation Equations

Counter-clockwise rotation by an angle θ



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P' = R.P}$$

Why does multiplying points by R rotate them?

• Think of the rows of R as a new coordinate system. Taking inner products of each points with these expresses that point in that coordinate system.

  • This means rows of R must be orthonormal vectors (orthogonal unit vectors).

• Think of what happens to the points (1,0) and (0,1). They go to (cos theta, -sin theta), and (sin theta, cos theta). They remain orthonormal, and rotate clockwise by theta.

  • Any other point, (a,b) can be thought of as a(1,0) + b(0,1). R(a(1,0)+b(0,1) = Ra(1,0) + Ra(0,1) = aR(1,0) + bR(0,1). So it's in the same position relative to the rotated coordinates that it was in before rotation relative to the x, y coordinates. That is, it's rotated.

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 → 4 elements

BUT! There is only 1 degree of freedom: θ

The 4 elements must satisfy the following constraints:

$$\mathbf{R} \cdot \mathbf{R}^{\mathbf{T}} = \mathbf{R}^{\mathbf{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Transformations can be composed

- Matrix multiplication is associative.
- Combine series of transformations into one matrix. *(example, whiteboard).*
- In general, the order matters. *(example, whiteboard).*
- 2D Rotations can be interchanged. Why?

Rotation and Translation

$$\begin{pmatrix} \cos q & -\sin q & tx \\ \sin q & \cos q & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

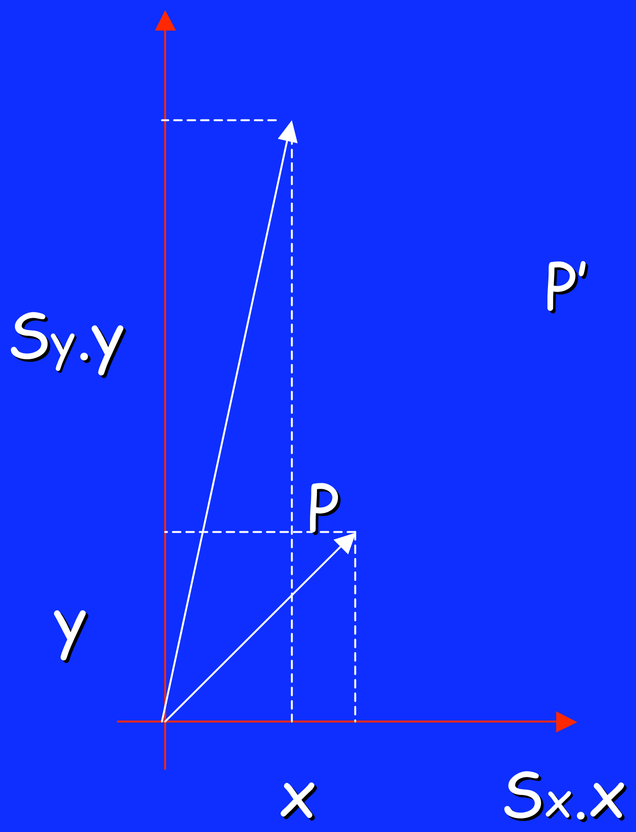
Rotation, Scaling and Translation

$$\begin{pmatrix} a & -b & tx \\ b & a & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotation about an arbitrary point

- Can translate to origin, rotate, translate back. *(example, whiteboard).*
- This is also rotation with one translation.
  – Intuitively, amount of rotation is same either way.
  – But a translation is added.

Stretching Equation



$$\mathbf{P} = (x, y)$$

$$\mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

Stretching = tilting and projecting (with weak perspective)

$$\mathbf{P'} \rightarrow \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = s_y \begin{bmatrix} \frac{s_x}{s_y} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

SVD

- For any matrix, $M = R_1 D R_2$
  - R_1, R_2 are rotation matrices
  - D is a diagonal matrix.
  - This decomposition is unique.
  - Efficient algorithms can compute this (in matlab, *svd*).

Linear Transformation

$$\mathbf{P'} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

SVD

$$= \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= s_y \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \frac{s_x}{s_y} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Affine Transformation

$$\mathbf{P'} \rightarrow \begin{bmatrix} a & b & tx \\ c & d & ty \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Simple 3D Rotation

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & . & . & . & x_n \\ y_1 & y_2 & & & & y_n \\ z_1 & z_2 & & & & z_n \end{pmatrix}$$

Rotation about z axis.

Rotates x,y coordinates. Leaves z coordinates fixed.

Full 3D Rotation

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

- Any rotation can be expressed as combination of three rotations about three axes.

$$RR^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rows (and columns) of *R* are orthonormal vectors.
- R has determinant 1 (not -1).

- Intuitively, it makes sense that 3D rotations can be expressed as 3 separate rotations about fixed axes. Rotations have 3 degrees of freedom; two describe an axis of rotation, and one the amount.

- Rotations preserve the length of a vector, and the angle between two vectors. Therefore, (1,0,0), (0,1,0), (0,0,1) must be orthonormal after rotation. After rotation, they are the three columns of R. So these columns must be orthonormal vectors for R to be a rotation. Similarly, if they are orthonormal vectors (with determinant 1) R will have the effect of rotating (1,0,0), (0,1,0), (0,0,1). Same reasoning as 2D tells us all other points rotate too.

  - Note if R has determinant -1, then R is a rotation plus a reflection.

3D Rotation + Translation

- Just like 2D case