Transformations

Vectors

- Ordered set of numbers: (1,2,3,4)
- Example: (*x*,*y*,*z*) coordinates of pt in space.

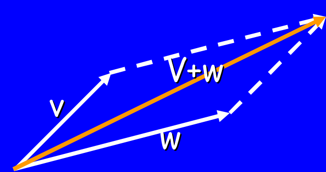
$$v = (x_1, x_2, \ldots, x_n)$$

$$\|v\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$\text{If } \|v\| = 1, v \text{ is a unit vector}$$

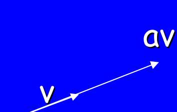
Vector Addition

$$\mathbf{v}+\mathbf{w}=(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2)$$

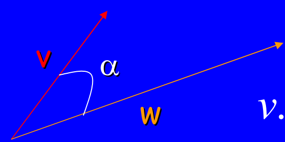


Scalar Product

$$a\mathbf{v}=a(x_1,x_2)=(ax_1,ax_2)$$



Inner (dot) Product



$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2.y_2$$

The inner product is a SCALAR!

$$v.w = (x_1, x_2).(y_1, y_2) = \| v \| \cdot \| w \| \cos \alpha$$

$$v.w = 0 \Leftrightarrow v \perp w$$

First, we note that if we scale a vector, we scale its inner product. That is, <sv,w> = s<v,w>. This follows pretty directly from the definition.

This means that the statement <v,w> = ||v|| ||w|| cos(alpha) is true if and only if it is the case that when v and w are unit vectors, <v,w> = cos(alpha), because:

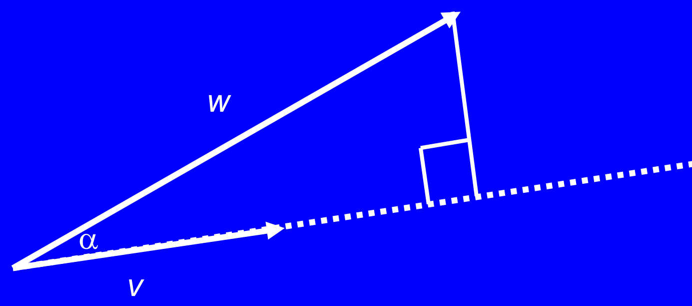
<v,w> = <(v/||v||),(w/||w||)> ||v|| ||w||. So from now on, we can assume that w, v are unit vector.

Then, as an example, we can consider the case where w = (1,0). It follows from the definition of cosine that <v,w> = cos(alpha). We can also see that taking <v,(1,0)> and <v,(0,1)> produces the (x,y) coordinates of v. That is, if (1,0) and (0,1) are an orthonormal basis, taking inner products with them gives the coordinates of a point relative to that basis. This is why the inner product is so useful. We just have to show that this is true for any orthonormal basis, not just (1,0) and (0,1).

How do we prove these properties of the inner product? Let's start with the fact that orthogonal vectors have 0 inner product. Suppose one vector is (x,y), and WLOG x,y>0. Then, if we rotate that by 90 degrees counterclockwise, we'll get (y, -x). Rotating the vector is just like rotating the coordinate system in the opposite direction. And (x,y)*(y,-x) = xy – yx = 0.

Next, note that if w1 + w2 = w, then v*w = v*(w1+w2) = v*w1 + v*w2. For any w, we can write it as the sum of w1+w2, where w1 is perpendicular to v, and w2 is in the same direction as v. So v*w1 = 0. v*w2 = ||w2||, since v*w2/||w2|| = 1. Then, if we just draw a picture, we can see that cos alpha = ||w2|| = v*w2 = v*w.

Inner product and direction



This tells us that if v is a unit vector (and w isn't) that

<v,w> = ||w|| cos(α). This is the *projection* of *w* onto *v.* It means that to get to *w*, we go a distance of <v,w> in the direction v, and then some distance in a direction orthogonal to v.

Matrices

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Sum:

$$C_{n\times m} = A_{n\times m} + B_{n\times m}$$

$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

Matrices

Product:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

A and B must have compatible dimensions

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Identity Matrix:

$$I = \begin{pmatrix} 1 & 0 & \ddots & 0 \\ 0 & 1 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & 1 \end{pmatrix} \qquad IA = AI = A$$

Matrices

Transpose:

$$C_{m\times n} = A^T{}_{n\times m}$$

$$c_{ij} = a_{ji}$$

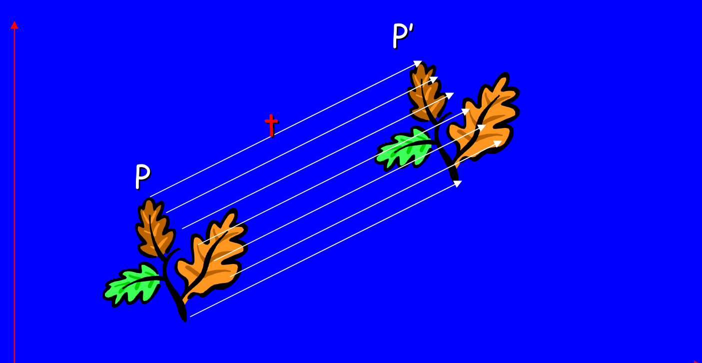
$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

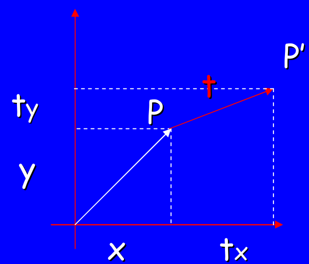
If $A^T = A$ A is symmetric

Euclidean transformations

2D Translation



2D Translation Equation

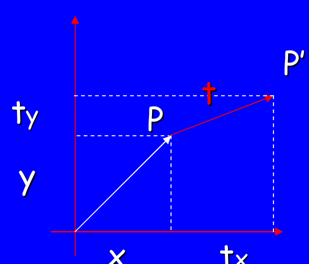


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = (x + t_x, y + t_y) = \mathbf{P} + \mathbf{t}$$

2D Translation using Matrices



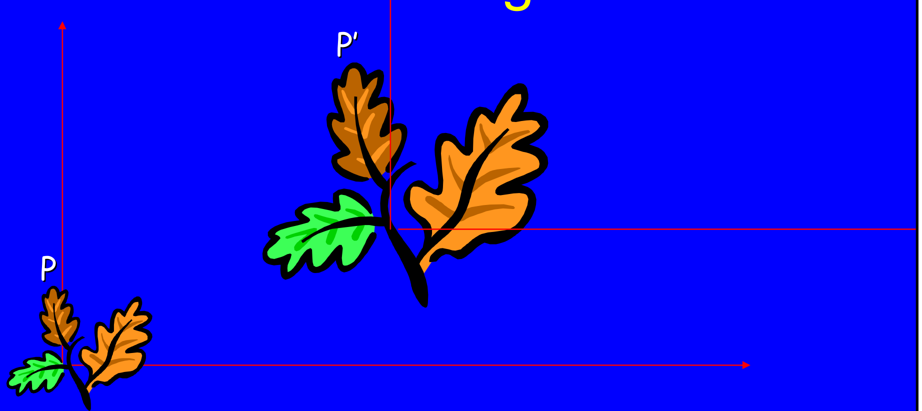
$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

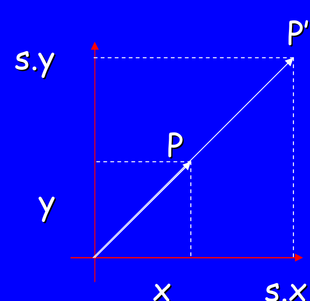
t P

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling



Scaling Equation



$$\mathbf{P} = (x, y)$$

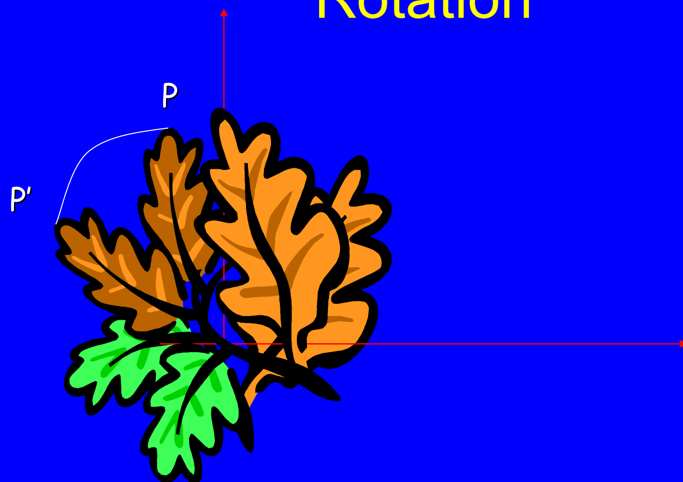
$$\mathbf{P'} = (sx, sy)$$

$$\mathbf{P'} = s \cdot \mathbf{P}$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} sx \\ sy \end{bmatrix} = \underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}}_{\mathbf{S}} . \begin{bmatrix} x \\ y \end{bmatrix}$$

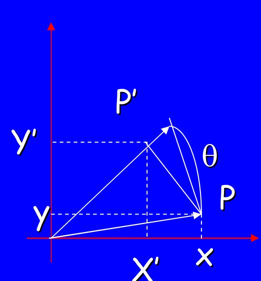
$$\mathbf{P'} = \mathbf{S} \cdot \mathbf{P}$$

Rotation



Rotation Equations

Counter-clockwise rotation by an angle θ



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R.P}$$

Why does multiplying points by R rotate them?

- Think of the rows of R as a new coordinate system. Taking inner products of each points with these expresses that point in that coordinate system.
  - This means rows of R must be orthonormal vectors (orthogonal unit vectors).
- Think of what happens to the points (1,0) and (0,1). They go to (cos theta, -sin theta), and (sin theta, cos theta). They remain orthonormal, and rotate clockwise by theta.
  - Any other point, (a,b) can be thought of as a(1,0) + b(0,1). R(a(1,0)+b(0,1) = Ra(1,0) + Ra(0,1) = aR(1,0) + bR(0,1). So it's in the same position relative to the rotated coordinates that it was in before rotation relative to the x, y coordinates. That is, it's rotated.

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 → 4 elements

BUT! There is only 1 degree of freedom: θ

The 4 elements must satisfy the following constraints:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Transformations can be composed

- Matrix multiplication is associative.
- Combine series of transformations into one matrix.
- In general, the order matters.
- 2D Rotations can be interchanged. Why?

Rotation and Translation

$$\begin{pmatrix} \cos\theta & -\sin\theta & tx \\ \sin\theta & \cos\theta & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

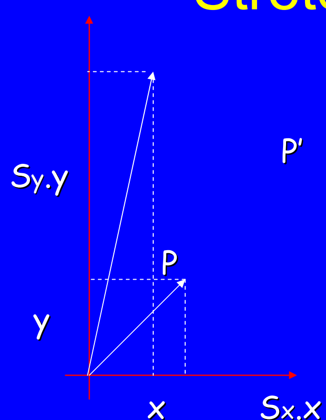
Rotation, Scaling and Translation

$$\begin{pmatrix} a & -b & tx \\ b & a & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Inverse of a rotation

- If R is a rotation, $RR^T = I$.
  - This is because the diagonals of RR^T are the magnitudes of the rows, which are all 1, because the rows are unit vectors giving directions.
  - The off-diagonals are the inner product of orthogonal unit vectors, which are zero.
- So the transpose of R is its inverse, a rotation of equal magnitude in the opposite direction.

Stretching Equation



$$\mathbf{P} = (x, y)$$

$$\mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

Stretching = tilting and projecting (with weak perspective)

$$\mathbf{P'} \rightarrow \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = s_y \begin{bmatrix} \frac{s_x}{s_y} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear Transformation

$$\mathbf{P'} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

SVD

$$= \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= s_y \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \frac{s_x}{s_y} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Affine Transformation

$$\mathbf{P'} \rightarrow \begin{bmatrix} a & b & tx \\ c & d & ty \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is equivalent to stretching by an arbitrary amount in an arbitrary direction, and translating.

Solving for Transformation

Let points in one image have coordinates : (x_i, y_i) and in the other image : (u_i, v_i). If they are related by an Affine Transformation :

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ 1 & 1 & 1 \end{pmatrix} = A \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = A$$

So we can solve for A with these correspondences.