Geometry
(Many slides adapted from Octavia Camps and Amitabh Varshney)

Much of material in Appendix A
Goals

- Represent points, lines and triangles as column vectors.
- Represent motion as matrices.
- Move geometric objects with matrix multiplication.
- Refresh memory about geometry and linear algebra
Vectors

- Ordered set of numbers: (1, 2, 3, 4)
- Example: (x, y, z) coordinates of pt in space.

\[ v = (x_1, x_2, \ldots, x_n) \]

\[ ||v|| = \sqrt{\sum_{i=1}^{n} x_i^2} \]

If \( ||v|| = 1 \), \( v \) is a unit vector.
Vector Addition

\[ \mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \]
Scalar Product

\[ a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2) \]
The inner product is a **SCALAR**!

\[ v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2 \]

\[ v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = \| v \| \cdot \| w \| \cos \alpha \]

\[ v \cdot w = 0 \iff v \perp w \]
First, we note that if we scale a vector, we scale its inner product. That is, $<sv,w> = s<v,w>$. This follows pretty directly from the definition.

So from now on, we can assume that $w$, and maybe $v$ are unit vectors. Then, as an example, we can consider the case where $w = (1,0)$. It follows from the definition of cosine that $<v,w> = ||v||\cos(\alpha)$. We can also see that taking $<v,(1,0)>$ and $<v,(0,1)>$ produces the $(x,y)$ coordinates of $v$. That is, if $(1,0)$ and $(0,1)$ are an orthonormal basis, taking inner products with them gives the coordinates of a point relative to that basis. This is why the inner product is so useful. We just have to show that this is true for any orthonormal basis, not just $(1,0)$ and $(0,1)$.

How do we prove these properties of the inner product? Let’s start with the fact that orthogonal vectors have 0 inner product. Suppose one vector is $(x,y)$, and WLOG $x,y>0$. Then, if we rotate that by 90 degrees counterclockwise, we’ll get $(y,-x)$. Rotating the vector is just like rotating the coordinate system in the opposite direction. And $(x,y)(y,-x) = xy - yx = 0$.

Next, note that $v^w = (v^w)/(||v|| ||w||) * ||v|| ||w||$. This means that if we can show that when $v$ and $w$ are unit vectors $v^w = \cos(\alpha)$, then it will follow that in general $v^w = ||v|| ||w|| \cos(\alpha)$. So suppose $v$ and $w$ are unit vectors.

Next, note that if $w1 + w2 = w$, then $v^w = v^w(w1+w2) = v^w1 + v^w2$. For any $w$, we can write it as the sum of $w1+w2$, where $w1$ is perpendicular to $v$, and $w2$ is in the same direction as $v$. So $v^w1 = 0$. $v^w2 = ||w2||$, since $v^w2/||w2|| = 1$. Then, if we just draw a picture, we can see that $\cos(\alpha) = ||w2|| = v^w2 = v^w$. 
Points

Using these facts, we can represent points. Note:

\[(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)\]

\[x = (x,y,z).(1,0,0) \quad y = (x,y,z).(0,1,0)\]

\[z = (x,y,z).(0,0,1)\]
Lines

• Line: $y = mx + a$

• Line: sum of a point and a vector
  $\mathbf{P} = \mathbf{P}_1 + \alpha \mathbf{d}$
  (where $\mathbf{d}$ is a column vector)

• Line: Affine sum of two points
  $\mathbf{P} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2$, where $\alpha_1 + \alpha_2 = 1$

  Line Segment: For $0 \leq \alpha_1, \alpha_2 \leq 1$, $\mathbf{P}$ lies between $\mathbf{P}_1$ and $\mathbf{P}_2$

• Line: set of points equidistant from the origin in the direction of a unit vector. $(a,b) \cdot (x,y) = -c$. 
Plane and Triangle

• Plane: sum of a point and two vectors
  \[ \mathbf{P} = \mathbf{P}_1 + \alpha \mathbf{u} + \beta \mathbf{v} \]

  Plane: set of points equidistant from origin in direction of a vector.

• Triangle: Affine sum of three points
  with \( \alpha_i \geq 0 \)
  \[ \mathbf{P} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \alpha_3 \mathbf{P}_3, \]
  where \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \)
  \( \mathbf{P} \) lies between \( \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \)
Generalizing ... 

Affine Sum of arbitrary number of points: Convex Hull 

\[ \mathbf{P} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \ldots + \alpha_n \mathbf{P}_n, \] 
where \( \alpha_1 + \alpha_2 + \ldots + \alpha_n = 1 \) and \( \alpha_i \geq 0 \)
Normal of a Plane

Plane: sum of a point and two vectors
\[ \mathbf{P} = \mathbf{P}_1 + \alpha \mathbf{u} + \beta \mathbf{v} \]
\[ \mathbf{P} - \mathbf{P}_1 = \alpha \mathbf{u} + \beta \mathbf{v} \]

If \( \mathbf{n} \) is orthogonal to \( \mathbf{u} \) and \( \mathbf{v} \) \( (\mathbf{n} = \mathbf{u} \times \mathbf{v}) \) :
\[ \mathbf{n}^T \cdot (\mathbf{P} - \mathbf{P}_1) = \alpha \mathbf{n}^T \cdot \mathbf{u} + \beta \mathbf{n}^T \cdot \mathbf{v} = 0 \]
Implicit Equation of a Plane

\[ \mathbf{n}^T \cdot (\mathbf{P} - \mathbf{P}_1) = 0 \]

Let \( \mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \)
\( \mathbf{P} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \)
\( \mathbf{P}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \)

Then, the equation of a plane becomes:

\[ a (x - x_1) + b (y - y_1) + c (z - z_1) = 0 \]

\[ a x + b y + c z + d = 0 \]

Thus, the coefficients of \( x, y, z \) in a plane equation define the normal.
Normal of a Triangle

Normal of the plane containing the triangle \((P_1, P_2, P_3)\):

\[
\vec{n} = (P_2 - P_1) \times (P_3 - P_1)
\]

- Normal pointing towards you
- Normal pointing away from you

• Models constructed with consistent ordering of triangle vertices:
  - all clockwise or all counter-clockwise.
• Usually normals point out of the model.
Normal of a Vertex in a Mesh

\[
\vec{n}_v = \frac{\vec{n}_1 + \vec{n}_2 + \ldots + \vec{n}_k}{k} = \frac{\sum \vec{n}_i}{k}
\]

= average of adjacent triangle normals

or better:

\[
\vec{n}_v = \frac{\sum (A_i \vec{n}_i)}{(k \sum (A_i))}
\]

= area-weighted average of adjacent triangle normals