Practice Midterm Solutions
CMSC 427

Sample Problems (Note, some of these problems may be a bit more involved than ones I'd ask on a time-limited exam).

1. Create a matrix that rotates points 90 degrees about the point (1,1).

We can do this by translating to the origin, rotating, and translating back. We get:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can verify that this is right by checking that it takes the point (1,1) (written (1,1,1) in homogenous coordinates) to the point (1,1), and that it takes the point (2,1) to (1,2).

2. What is the distance from (3,2) to (7,5) in the direction (1,2)?

To do this we need to find the inner product between a vector from (3,2) to (7,5) and a unit vector in the direction (1,2). The first vector is (4,3). The unit vector we need is (1,2)/sqrt(5). This inner product is 10/sqrt(5)=2sqrt(5)

3. Provide any two rows of a 4x4 matrix that will transform 3D points as they would appear in the coordinate system of a viewer centered at (3,2,0) facing in the direction (1,2,3). The new *z* coordinate should describe the distance from the viewer to a point, in the direction that they are viewing it.

We are changing the origin and the viewing direction. We can do this by combining two matrices. If we just wanted to change origins, all we would need to do is create a 3D translation matrix that takes (3,2,0) to the origin. This would be:

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

However, we also want to change the viewing direction. To do this, we'll create a matrix in which the first three elements of the third row represent our new z direction. This direction is represented by the unit vector (1,2,3)/sqrt(14). So, the matrix we want will look like:

$$\begin{pmatrix} & & & \\ & & & \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} & & & \\ & & & \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{-7}{\sqrt{14}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that to get the bottom two rows we never had to come up with the top two rows of the first matrix on the left.

4. Suppose you have a camera with a focal point at (0,0,0) and an image plane of z=1. Give an example of a right triangle in 3D that will also appear in the image as a right triangle, assuming perspective projection. Do the same thing for orthographic projection.

Perspective projection scales points by the inverse of their distance. So, if we pick points that have the same z coordinate, they will all be scaled the same amount. So, for example, if we take a triangle with vertices at (0,0,2) (2,0,2), (0, 2, 2), this will appear in images with coordinates (0,0), (1,0), (0,1), and still be a right triangle.

With orthographic projection, the same triangle will appear as (0,0), (2,0), (0,2), also a right triangle.

5. Prove that parallel lines in the world do not always appear as parallel lines with perspective projection.

We just need an example to show this. Let's consider lines that lie in the y=-5 plane, with our standard perspective camera. Consider a line with the equation z = mx+b. This contains the points (0,-5,b) and (1,-5, m+b). These appear in the image as points (0, -5/b) and (1/(m+b), -5/(m+b)). This line has a slope of (-5/(m+b) + 5/b)/(1/(m+b)) = -5 + 5(m+b)/b = 5m/b. Clearly this slope depends not just on m, the slope of the original line, but also on b. Two lines with the same value for m and different values for b will be parallel in the world, but will have different slopes in the image.

6. Explain how you would adapt the algorithm we learned that discretizes lines to discretely represent the boundary of a circle. What pixels would be filled in to represent a circle centered at (20,20) with a radius of 5?

First, let's consider the simpler, naïve algorithm. We cannot just increment x from 15 to 25, and solve for two y values for each x, because in some parts of the circle y changes more rapidly than x, so we would get a circle with gaps. Instead, we can divide the circle into regions where either x or y is changing more rapidly. For example, at an angle between pi/4 and 0, y is changing more rapidly. This occurs in the region of the circle starting at (20+5/sqrt(2), 20+5/sqrt(2)) and ending at (25,20). Rounding off, this means we must consider y values of 24, 23, 22, 21, 20. For these, the corresponding x values are 23, 24, 25,25,25, so we fill in points (23,24), (24,23), (25,22), (25,21), (25,20).

We can try to adapt DDA to this problem by noticing that the tangent to a circle at a boundary point is perpendicular to a vector from the center to the boundary point. So if we start at the point (25,20), a vector from (20,20) to (25,20) is (5,0), and the tangent at that point is 0/5. If we change y by 1, we should not change x at all, and we find the point (25,21). At this point the slope will be -1/5, so the next point on the circle will be (24 4/5, 22). Rounding this off, we would fill in the pixel (25,22). The slope at this point would be -2/(4+4/5) = -5/12. So the next point on the circle will be (24 23/60, 23), which rounds to (24,23). Continuing this way brings us to (23.7, 24), rounded to (24,24). Results are a bit less accurate. Is this worth it?

7. Suppose, for the purposes of this problem, that Philadelphia is 90 miles north-east of Washington, and that Baltimore is 30 miles northeast of Washington.
    a. If it is 40 degrees in Washington, and 31 degrees in Philadelphia, use linear interpolation to estimate the temperature in Baltimore.

Baltimore is 1/3 of the way from Washington to Philadelphia. So we can express the position of Baltimore as (2/3)*Washington + (1/3)*Philadelphia. If we express the temperature in the same way, we get: (2/3)*40 + (1/3)*31 = 37 degrees.

    b. Suppose Harrisburg is 60 miles north of Washington, and it is 24 degrees there. Use linear interpolation to estimate the temperature at a location that is 30 miles north of Washington and 15 miles east of it.

First, let's consider a point that is 30 miles north of Washington. This is halfway in between Washington and Harrisburg, and so linear interpolation predicts a temperature of 32 degrees. Next, we can interpolate between Washington and Philadelphia to get the temperature 30 miles north and 30 miles east of Washington. This is sqrt(18)/9 = .4714 of the way from Washington to Philly. Interpolating, we find the temperature here to be: 35.76 degrees. The point we want is halfway between these two places, so averaging gives its temperature as: 33.8 degrees.

    c. What do these problems have to do with graphics?

This is exactly the way we determine the z value of every point in a triangle, given the value at the corners.

8. Consider a triangle with corners at (2,2,3) (3,3,4), (3,1,3).
    a. What is the normal vector to this triangle?

The normal should be orthogonal to the vectors from the first point to the second and third ones. These vectors are: (1,1,1) and (1,-1,0). Suppose the vector (a,b,c) is orthogonal to both these. We have a + b + c = 0, and a-b = 0 (which implies a = b). So, if we pick c = 2, we get a = b = -1, or (-1, -1, 2). We should normalize this by dividing the value by sqrt(6).

    b. What is an equation for the plane that this triangle lies in?

We can say that the inner product between the normal vector and any point on the plane is a constant, d. So we get:

$-x - y + 2z = d.$

Plugging in (2,2,3), we get: $-2 - 2 + 6 = d$, so $d = 2$, giving:

$-x - y + 2z = 2.$

We can verify that this equation holds for all vertices of the triangle.

c. What test would the painter's algorithm use to decide paint this triangle before a triangle with vertices at (2,3,2), (3,2,2), (3,3,3.5)?

The min z value of the first triangle is not greater than the max z value of the second, so this doesn't work. If we take the inner product of the first's normal with each point on the new triangle, we get:

$<(-1,-1,2), (2,3,2)> = -1 < 2$
$<(-1,-1,2), (3,2,2)> = -3 < 2$
$<(-1,-1,2), (3,3,3.5)> = 1 < 2.$

Note that for the origin:

$<(-1,-1,2),(0,0,0)> = 0 < 2.$

These are on the same side of the first triangle's plane as the origin, so the second triangle is in front of the first, and we can render it first.