Tranformations

Some slides adapted from Octavia Camps

Matrices

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Sum:

$$C_{n\times m} = A_{n\times m} + B_{n\times m}$$

$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

Matrices

Product:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

A and B must have compatible dimensions

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Identity Matrix:

$$I = \begin{pmatrix} 1 & 0 & \ddots & 0 \\ 0 & 1 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & 1 \end{pmatrix} \qquad IA = AI = A$$

Matrices

- Associative T*(U*(V*p)) = (T*U*V)*p
- Distributive T*(u+v) = T*v + T*v

Matrices

Transpose:

$$C_{m\times n} = A^T{}_{n\times m}$$

$$c_{ij} = a_{ji}$$

$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

If $A^T = A$ A is symmetric

Matrices

Determinant: *A must be square*

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Matrices

Inverse:

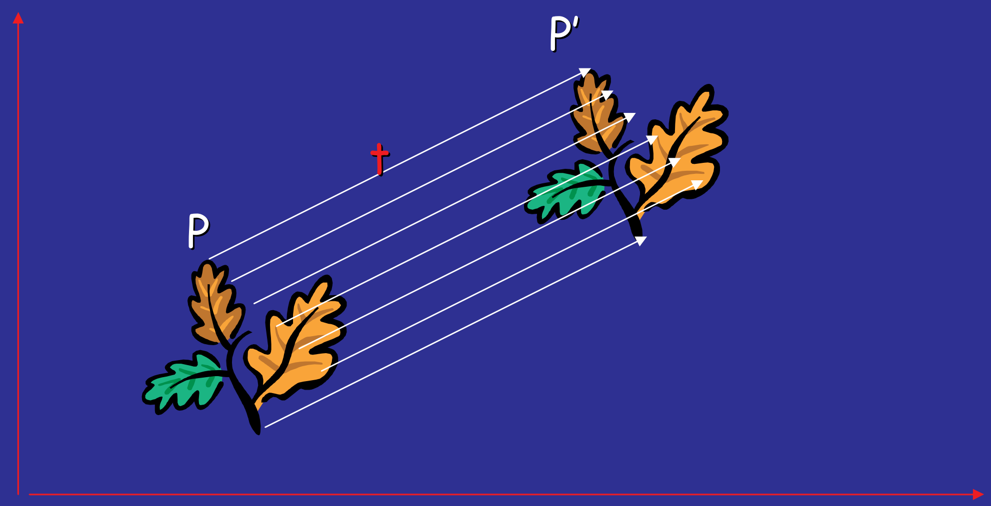
A must be square

$$A_{n\times n} A^{-1}{}_{n\times n} = A^{-1}{}_{n\times n} A_{n\times n} = I$$

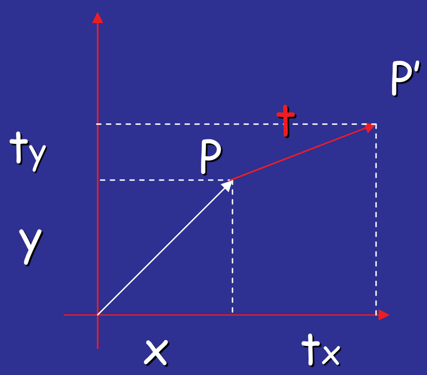
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Euclidean transformations

2D Translation



2D Translation Equation

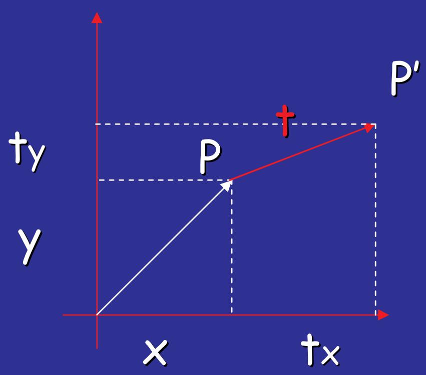


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = (x + t_x, y + t_y) = \mathbf{P} + \mathbf{t}$$

2D Translation using Matrices

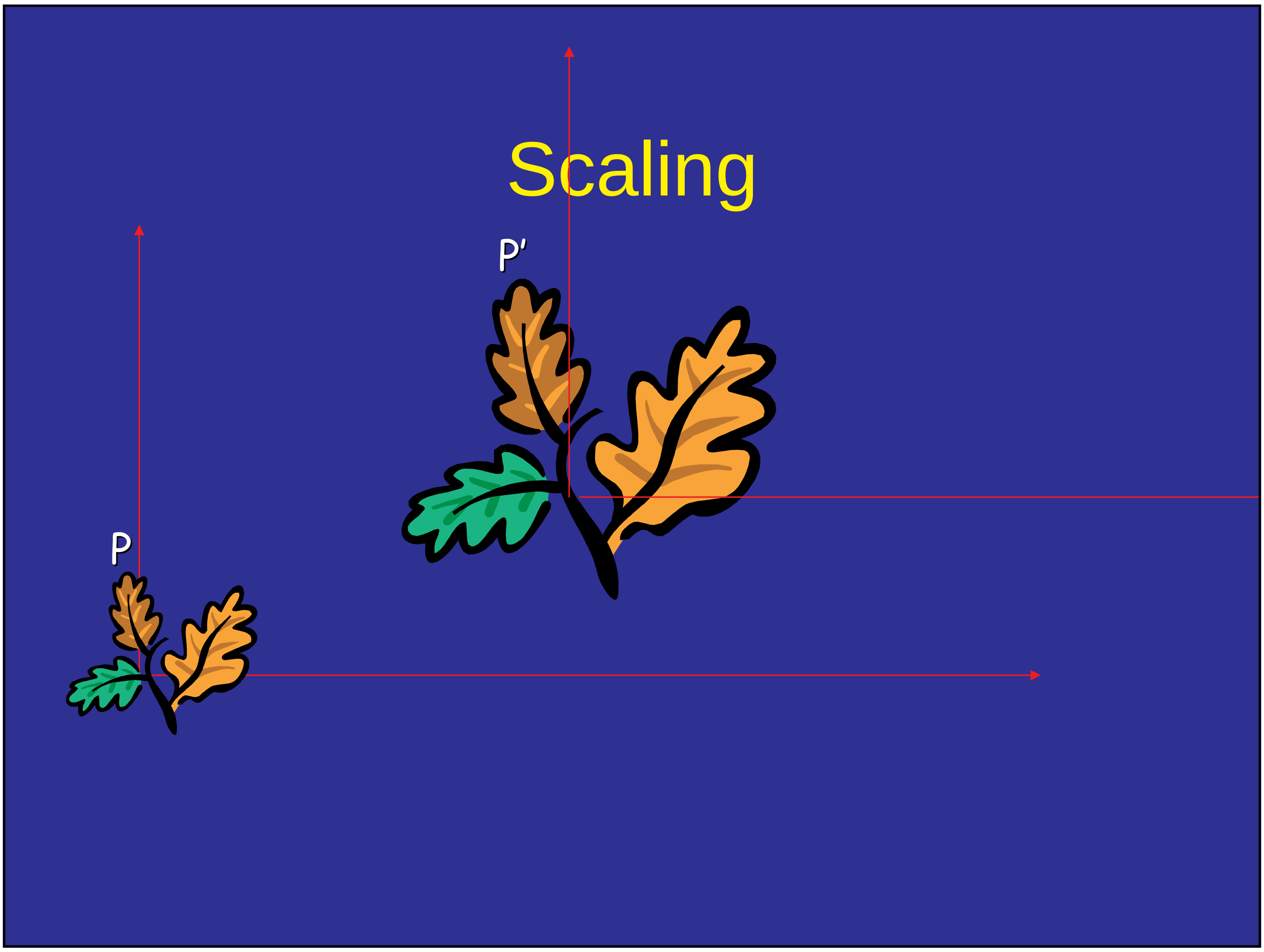


$$\mathbf{P} = (x, y)$$

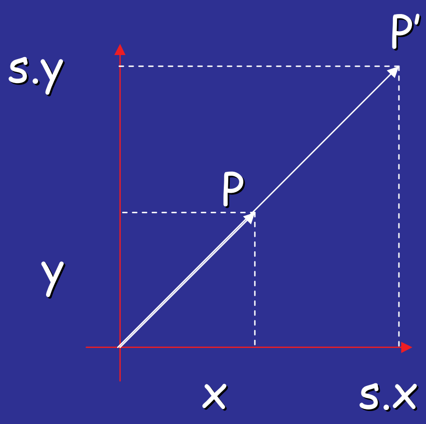
$$\mathbf{t} = (t_x, t_y)$$

t P

$$\mathbf{P'} \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$


Scaling
P'
P

Scaling Equation



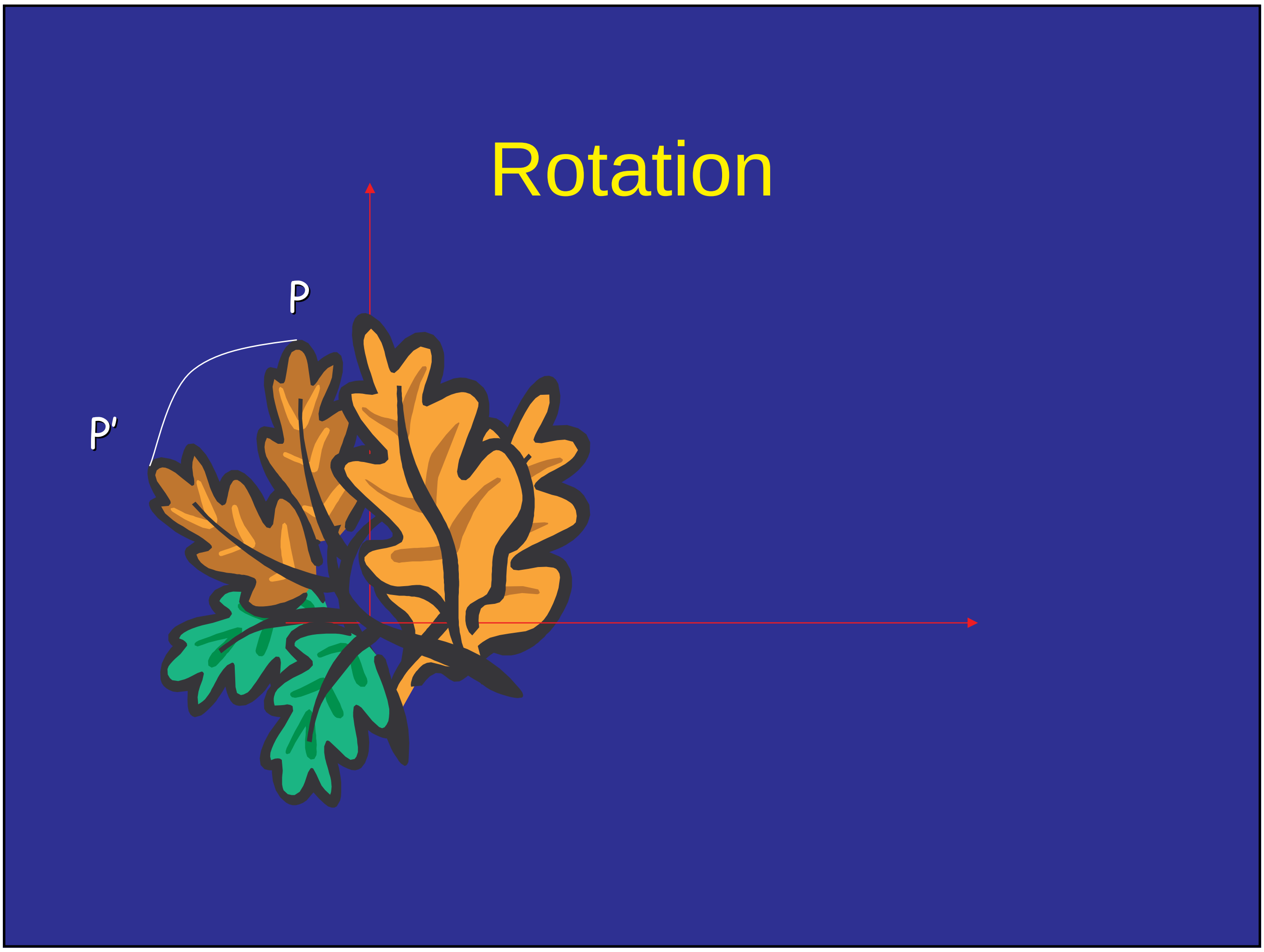
$$\mathbf{P} = (x, y)$$

$$\mathbf{P}' = (sx, sy)$$

$$\mathbf{P}' = s \cdot \mathbf{P}$$

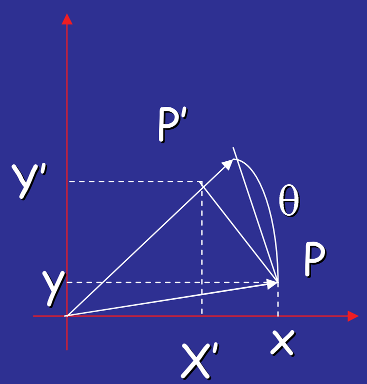
$$\mathbf{P}' \rightarrow \begin{bmatrix} sx \\ sy \end{bmatrix} = \underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$


Rotation
P
P'

Rotation Equations

Counter-clockwise rotation by an angle θ



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P' = R.P}$$

Why does multiplying points by R rotate them?

- Think of the rows of R as a new coordinate system. Taking inner products of each points with these expresses that point in that coordinate system.
  - This means rows of R must be orthonormal vectors (orthogonal unit vectors).
- Think of what happens to the points (1,0) and (0,1). They go to (cos theta, -sin theta), and (sin theta, cos theta). They remain orthonormal, and rotate clockwise by theta.
  - Any other point, (a,b) can be thought of as a(1,0) + b(0,1). R(a(1,0)+b(0,1) = Ra(1,0) + Ra(0,1) = aR(1,0) + bR(0,1). So it's in the same position relative to the rotated coordinates that it was in before rotation relative to the x, y coordinates. That is, it's rotated.

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 → 4 elements

BUT! There is only 1 degree of freedom: θ

The 4 elements must satisfy the following constraints:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Transformations can be composed

- Matrix multiplication is associative.
- Combine series of transformations into one matrix. *(example, whiteboard).*
- In general, the order matters. *(example, whiteboard).*
- 2D Rotations can be interchanged. Why?

Rotation and Translation

$$\begin{pmatrix} \cos\theta & -\sin\theta & tx \\ \sin\theta & \cos\theta & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotation, Scaling and Translation

$$\begin{pmatrix} a & -b & tx \\ b & a & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

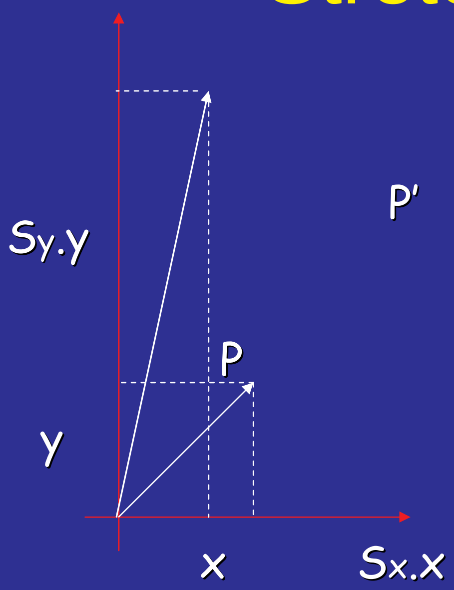
Rotation about an arbitrary point

- Can translate to origin, rotate, translate back. *(example, whiteboard).*
- This is also rotation with one translation.
  - Intuitively, amount of rotation is same either way.
  - But a translation is added.

Inverse of a rotation

- If R is a rotation, $RR^T = I$.
  - This is because the diagonals of RR^T are the magnitudes of the rows, which are all 1, because the rows are unit vectors giving directions.
  - The off-diagonals are the inner product of orthogonal unit vectors, which are zero.
- So the transpose of R is its inverse, a rotation of equal magnitude in the opposite direction.

Stretching Equation



$$\mathbf{P} = (x, y)$$

$$\mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

Stretching = tilting and projecting (with weak perspective)

$$\mathbf{P'} \rightarrow \begin{bmatrix} s_x x \\ s_y y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = s_y \begin{bmatrix} \frac{s_x}{s_y} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear Transformation

$$\mathbf{P}' \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

SVD

$$= \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= s_y \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \frac{s_x}{s_y} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Affine Transformation

$$\mathbf{P}' \rightarrow \begin{bmatrix} a & b & tx \\ c & d & ty \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Viewing Position

- Express world in new coordinate system.
- If origins same, this is done by taking inner product with new coordinates.
- Otherwise, we must translate.

Suppose, for example, we want to have the y axis show how we are facing. We want to be at (7,3), facing in direction (ct,st). The x axis must be orthogonal, (-st,ct). If we want to express (x,y) in this coordinate frame, we need to take: (ct,st)*(x-7,y-3), and (-st,ct)*(x-7,y-3). This is done by multiplying by matrix with rows (-st,ct) and (ct,st)

Simple 3D Rotation

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & . & . & . & x_n \\ y_1 & y_2 & & & & y_n \\ z_1 & z_2 & & & & z_n \end{pmatrix}$$

Rotation about z axis.

Rotates x,y coordinates. Leaves z coordinates fixed.

Full 3D Rotation

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

- Any rotation can be expressed as combination of three rotations about three axes.

$$RR^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rows (and columns) of *R* are orthonormal vectors.
- R has determinant 1 (not -1).

• Intuitively, it makes sense that 3D rotations can be expressed as 3 separate rotations about fixed axes. Rotations have 3 degrees of freedom; two describe an axis of rotation, and one the amount.

• Rotations preserve the length of a vector, and the angle between two vectors. Therefore, (1,0,0), (0,1,0), (0,0,1) must be orthonormal after rotation. After rotation, they are the three columns of R. So these columns must be orthonormal vectors for R to be a rotation. Similarly, if they are orthonormal vectors (with determinant 1) R will have the effect of rotating (1,0,0), (0,1,0), (0,0,1). Same reasoning as 2D tells us all other points rotate too.

  • Note if R has determinant -1, then R is a rotation plus a reflection.

3D Rotation + Translation

- Just like 2D case

Transformation of lines/normals

- 2D. Line is set of points (x,y) for which (x,y).$(ab)^T$=0. Suppose we rotate points by R. We want a matrix, T, so that:

R*(x,y) .T

3D Viewing Position

- Rows of rotation matrix correspond to new coordinate axis.

Rotation about a known axis

- Suppose we want to rotate about u.
- Find R so that u will be the new z axis.
  - u is third row of R.
  - Second row is anything orthogonal to u.
  - Third row is cross-product of first two.
  - Make sure matrix has determinant 1.