"When images of illuminated objects ... penetrate through a small hole into a very dark room ... you will see [on the opposite wall] these objects in their proper form and color, reduced in size ... in a reversed position, owing to the intersection of the rays".

*Da Vinci*

http://www.acmi.net.au/AIC/CAMERA_OBSCURA.html (Russell Naughton)
• Used to observe eclipses (eg., Bacon, 1214-1294)
• By artists (eg., Vermeer).
Jetty at Margate England, 1898.

http://brightbytes.com/cosite/collection2.html (Jack and Beverly Wilgus)
Cameras

- First photograph due to Niepce
- First on record 1822
Pinhole cameras

- Abstract camera model - box with a small hole in it
- Pinhole cameras work in practice

(Forsyth & Ponce)
The equation of projection

(Forsyth & Ponce)
The equation of projection

- Cartesian coordinates:
  - We have, by similar triangles, that $(x, y, z) \rightarrow \left( f \frac{x}{z}, f \frac{y}{z}, f \right)$
  - Ignore the third coordinate, and get $(x, y, z) \rightarrow \left( f \frac{x}{z}, f \frac{y}{z} \right)$
Distant objects are smaller

(Forsyth & Ponce)
For example, consider one line segment from \((x, 0, z)\) to \((x, y, z)\), and another from \((x, 0, 2z)\) to \((x, y, 2z)\). These are the same length.

These project in the image to a line from \((fx/z, 0)\) to \((fx/z, fy/z)\) and from \((fx/z, 0)\) to \((fx/2z, fy/2z)\), where we can rewrite the last point as: \((1/2)(fx/z, fy/z)\). The second line is half as long as the first.
Parallel lines meet

Common to draw image plane *in front* of the focal point. Moving the image plane merely scales the image.

(Forsyth & Ponce)
Vanishing points

- Each set of parallel lines meets at a different point
  - The *vanishing point* for this direction
- Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  - The line is called the *horizon* for that plane
For example, let’s consider a line on the floor. We describe the floor with an equation like: \( y = -1 \). A line on the floor is the intersection of that equation with \( x = az + b \). Or, we can describe a line on the floor as: \( (a, -1, b) + t(c, 0, d) \) (Why is this correct, and why does it have more parameters than the first way?)

As a line gets far away, \( z \to \infty \). If \((x,-1,z)\) is a point on this line, its image is \( f(x/z,-1/z) \). As \( z \to \infty \), \(-1/z \to 0\). What about \( x/z? \) \( x/z = (az+b)/z = a + b/z \to a \). So a point on the line appears at: \((a,0)\). Notice this only depends on the slope of the line \( x = az + b \), not on \( b \). So two lines with the same slope have images that meet at the same point, \((a,0)\), which is on the horizon.
Properties of Projection

• Points project to points
• Lines project to lines
• Planes project to the whole image
• Angles are not preserved
• Degenerate cases
  – Line through focal point projects to a point.
  – Plane through focal point projects to line
  – Plane perpendicular to image plane projects to part of the image (with horizon).
Weak perspective (scaled orthographic projection)

- Issue
  - perspective effects, but not over the scale of individual objects
  - collect points into a group at about the same depth, then divide each point by the depth of its group

(Forsyth & Ponce)
The Equation of Weak Perspective

\((x, y, z) \rightarrow s(x, y)\)

- \(s\) is constant for all points.
- Parallel lines no longer converge, they remain parallel.
Pros and Cons of These Models

• Weak perspective much simpler math.
  – Accurate when object is small and distant.
  – Most useful for recognition.

• Pinhole perspective much more accurate for scenes.
  – Used in structure from motion.

• When accuracy really matters, must model real cameras.
Projective Transformations

• Mapping from plane to plane.
• Form a group.
  – They can be composed
  – They have inverses.
  – Projective transformations equivalent to set of images of images.
Linear Representation of Projective transformation

• Problem: $x/z, y/z$ is non-linear.
• Answer: Homogenous coordinates.
  – By definition, $(x,y,w) = (kx,ky,kw)$
  – So $(x,y,w) = (x/w, y/w, 1)$
3D rigid motion + projection (because of homogenous coordinates, there is no projection)

\[
\begin{pmatrix}
1 & 0 & 0 & t_x \\
0 & 1 & 0 & t_y \\
0 & 0 & 1 & t_z \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
r_{1,1} & r_{1,2} & r_{1,3} & 0 \\
r_{2,1} & r_{2,2} & r_{2,3} & 0 \\
r_{3,1} & r_{3,2} & r_{3,3} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
1 \\
\end{pmatrix}
\equiv
\begin{pmatrix}
r_{1,1} & r_{1,2} & r_{1,3} & t_x \\
r_{2,1} & r_{2,2} & r_{2,3} & t_y \\
r_{3,1} & r_{3,2} & r_{3,3} & t_z \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
1 \\
\end{pmatrix}
\]
For Planar Objects

\[
\begin{pmatrix}
    r_{1,1} & r_{1,2} & r_{1,3} & t_x \\
    r_{2,1} & r_{2,2} & r_{2,3} & t_y \\
    r_{3,1} & r_{3,2} & r_{3,3} & t_z \\
\end{pmatrix}
\begin{pmatrix}
    x \\
    y \\
    0 \\
\end{pmatrix}
\equiv
\begin{pmatrix}
    r_{1,1} & r_{1,2} & t_x \\
    r_{2,1} & r_{2,2} & t_y \\
    r_{3,1} & r_{3,2} & t_z \\
\end{pmatrix}
\begin{pmatrix}
    x \\
    y \\
    1 \\
\end{pmatrix}
\]

The first two columns on right are orthonormal. Scale is irrelevant. So there are 6 degrees of freedom.

We ignore constraints to get 8. This is called a projective transformation. By convention we scale the matrix so the lower right right value is 1.
Points at Infinity

• Note that we can have points like (1,1,0).
• If we thought of the third coordinate as za, and the \((x,y)\) coordinates as \((1/0,1/0)\), this point would be at infinity.
• This will make sense. These are points that are infinitely far away, in the x or y direction. We need to represent them because these points can become visible, eg., the horizon.
Example: Suppose we take a fronto-parallel planar surface, at z = 0, and transform it so that it is the
ground plane, using a transformation like:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 1
\end{pmatrix}
\]

This takes a point like (x,y,1) and transforms it to (x, -1, y+1). In 3D we can think of this as
transforming it to lie on the y = -1 plane.

Now consider a line, of the form (x,y,1) + (tu,tv,0) = (x+tu,y+tv,1), for all values of t. Applying this
transformation gives: (x+tu, -1, y+tv+1). As t -> infinity, this converges to (u/v, 0, 1). So a point at
infinity is being mapped to a point on the horizon, which has y = 0.

Another way to think of this is to consider the point (u,v,0). This is a point at infinity. Applying the
transformation to this point produces: (u, 0, v), which is the same as (u/v, 0, 1). The point (u,v,0) is the
same point as (x+tu,-1,y+tv+1) in the limit as t goes to infinity. And this transformation is mapping this
point to a point that is not at infinity, but that is visible.
Solving for Projective Transformation w/ Points

• One point: \( P_x = ku \), where \( x \) is point in one image, \( u \) in second image.
• 3 Linear equations with 8 unknowns for \( P \) and one for \( k \).
• Each new points provides 3 equations, one unknown.
• 4 points means 12 equations, 12 unknowns.
Lines

- We can represent a line in projective space as \( L = (a, b, c)^T \).
- Then the equation for a point to lie on a line is \( L^T p = 0 \).
- Given two lines, \( L \) and \( L' \), we can find their intersection by solving for \( L^T p = 0, L'^T p = 0 \).
- These are two linear equations, so all pairs of lines intersect. (unless \( L = L' \), which are the same line).
- So parallel lines intersect. Lines are parallel but different when \( k(a, b) = (a', b') \), with \( kc \neq c' \). So \( ax + by + cw = 0 = kax + kby + c'w = 0 \). So \( w = 0 \). Intersection point is at infinity.
Cameras with Lenses

(Forsyth & Ponce)
CCD Cameras

http://huizen.ddsw.nl/bewoners/maan/imaging/camera/ccd1.gif