Fourier Transform

1 Introduction

We will look at the Fourier transform and Wavelet transform as ways of representing signals and images. They are relevant to our class for a couple of reasons. First, it gives us a chance to review the idea of an orthogonal change of basis, which is central to any linear representation of data. For finite spaces, this change of basis is just a rotation of the original basis; for infinite dimensional spaces like function spaces, this is also true, but may seem a little trickier. Second, by dividing images and signals in terms of their frequency, the Fourier basis leads naturally to a dimensionality reduction method. One can think of this as dimensionality reduction, not in a way that is dependent on the data, like PCA, but using prior knowledge. We rotate our coordinate system so that some dimensions are known to contain less important or noisier components of the data, and then eliminate (project over) those dimensions. Third, the wavelet basis leads to a representation in which piecewise smooth signals tend to be sparse, which leads us to sparse representations.

2 Fourier Transform

We'll start with the Fourier Transform. This is one effective way of representing images. To see this, we first ask what properties a good image representation should have. But first, we back up and look at familiar representations.

How do we represent points in 2d? Using their x and y coordinates. What this really means is that we write any point as a linear combination of two vectors $(1,0)$ and $(0,1)$.

$$(x, y) = x(1, 0) + y(0, 1).$$

These vectors form an orthonormal basis for the plane. That means they are orthogonal $\langle(1,0),(0,1)\rangle = 0$, and of unit magnitude. Why is an orthonormal basis a good representation? There are many reasons, but two are:

1. Projection. To find the x coordinate, we take $\langle(x,y),(1,0)\rangle$.
2. Pythagorean theorem. $\|(x,y)\|^2 = x^2 + y^2$.

The Fourier series will provide an orthonormal basis for images.

2.1 Image Representations:

To simplify, I'll do everything in terms of a 1D function $f(t)$, but all this extends to 2D images. We'll start by considering periodic functions that go from 0 to 2π, which turn out to be easier. The main thing I'll do to make things easier, though, is to be very sloppy mathematically, skipping precise conditions and giving intuitions instead of complete proofs. For real math, refer to one of the references.

We can think of our ordinary representation of functions as like the sum of delta functions. Recall a delta function, $d_s(t)$ is 0 for $t \neq s$, infinite at $t = s$, but $\int d_s(t)dt = 1$.

$$\begin{aligned} f(t) &= \int \left(d_s(t) \int d_s(t) f(t) dt \right) ds \\ &= \int f(s) d_s(t) ds \end{aligned}$$

Note that in the first line, we say that $f(t)$ is a linear combination of scaled delta functions. The coefficient of each delta function is found by taking the inner product between that function and f (we take the inner product between two functions by multiplying them together and integrating. This is the continuous version of taking the inner product of vectors by multiplying corresponding coefficients and summing). That is, just as we find the x coordinate of a point, p, by taking the inner product between p and $(1,0)$, we take the $d_s(t)$ coordinate of f by taking the inner product between $d_s(t)$ and f.

The second line gives $f(t)$ as an infinite sum of scaled delta functions. In a sense, these form an orthonormal basis, but they are uncountable, which is awkward. For example, any finite subset of them provide a poor approximation to f (or any countable subset for that matter). We can do much better.

Another representation is familiar from the Riemann integral. Define a series of step functions that fill the interval. That is, let:

$$g_{k,i}(t) = \sqrt{\frac{k}{2\pi}} \quad \texttt{for} \quad \frac{2\pi(i-1)}{k} <= t <= \frac{2\pi i}{k}$$

with $g_{k,i}(t) = 0$ for other values of t.

Then $g_k = g_{k,1}, g_{k,k}$ provides a finite, orthonormal basis for piecewise constant functions. We can approximate f as a linear combination of these.

$$f(t) \approx \sum_{i=1}^{k} a_i g_{k,i}$$

In the limit, this approximation becomes arbitrarily good. The Riemann integral computes the sum of the area of functions, in the limit.

If we take the union of all g_k we get a countable set of functions that span a linear subspace that gets arbitrarily close to any $f(t)$. So we can represent any $f(t)$ as a countable sum.

Question: $f(t)$ has an uncountable number of degrees of freedom. How can we represent it with a countable representation? Answer: We make some continuity assumptions. These imply that if we know $f(t)$ at every rational number, we can determine it at every real number, so they reduce the degrees of freedom of $f(t)$ to the countable.

However, the union of all g_k do not provide an orthonormal basis, since they are redundant and non-orthogonal. Later, we'll see how a sort of variation on this basis gives us an orthonormal wavelet basis. For now, we look at the fourier series.

2.2 Fourier Series:

The following functions provide an orthonormal basis for functions:

$$\sqrt{\frac{1}{2\pi}}, \qquad \frac{\cos(kt)}{\sqrt{\pi}}, \quad \frac{\sin(kt)}{\sqrt{\pi}} \ \texttt{for} \ k = 1, 2,$$

We can show these are unit vectors by integrating them from 0 to 2π. We can show they are orthonormal by symmetry (or explicit integration).

To show they form a basis, we must show that, for any function with appropriate smoothness conditions,

$$f(t) = a_0 + \sum_{1}^{\infty} a_k \cos(kt) + b_k \sin(kt)$$

That is,

$$\lim_{N \to \infty} \int (f(t) - a_0 - \sum_{1}^{N} a_k \cos(kt) + b_k \sin(kt))^2 = 0$$

Why is this true? The intuition is that we first create a series of delta-like functions like $\cos^{2n}(t/2)$. We show that any function can be represented by these in the same way we did with the series g_k. Then show these are linear combinations of $\cos(kt)$, $\sin(kt)$ using trigonometric identities such as $\sin(t/2) = \sqrt{(((1-\cos(t))/2))}$ and $\sin^2 t = (1-\cos(2t))/2$.

Notation. We can combine $\sin$ and $\cos$ into one complex number. $e^{ikt} = cos(kt) + isin(kt)$. We let k go from $-\infty$ to ∞. This gives us a basis for complex functions (which we don't care about right now) and somewhat simpler notation. By proper choice of a_k we can get the real fourier series out of this. This is because if f is a real function, then $c_k =< f, e^{ikt} >=< f, cos(kt) > +i < f, sin(kt) >$ while $c_{-k} =< f, e^{-ikt} >=< f, \cos(kt) > -i < f, \sin(kt) >$. So $c_k e^{ikt} + c_{-k} e^{-ikt} = a_k \cos(kt) + b_k \sin(kt)$.

$< f, e^{ikt} >$ and $< f, e^{-ikt} >$ will be complex conjugates.

Since the coefficients of an imaginary fourier series are complex, it's often useful to discuss its two components as phase and magnitude.

$$F(t) = R(t) + iI(t)$$

Magnitude is $(R(u)^2 + I(u)^2)^{1/2}$, phase is $tan^{-1} I(u)/R(u)$.

2.3 Implications of the orthonormality of the fourier series

As with an orthonormal basis for vectors, the orthonormality of the fourier series means that we can use projection and (a generalization of) the Pythagorean theorem.

We define the inner product between functions just the same way we do between vectors. Multiplying and summing. So, for example:

$$< f(t), sin(t) >= \int f(t) sin(t) dt$$

So we have:

$$a_0 = (1/\pi) < f(t), 1 >= (1/2\pi) \int f(t)dt.$$

This is called the DC component of f. And:

$$a_k = (1/\pi) < f(t), \cos(kt) >= (1/\pi) \int f(t)\cos(kt)dt$$

$$b_k = (1/\pi) < f(t), \sin(kt) >= (1/\pi) \int f(t)\sin(kt)dt$$

And then:

$$f(t) = a_0/2 + \sum a_k \cos(kt) + \sum b_k \sin(kt)$$

The analog to the Pythagorean theorem is called Parsevaal's theorem. This is:

$$\int f^2(t) = (\pi/2)a_0^2 + \pi(\sum a_i^2 + b_i^2).$$

And if we approximate a function with a finite number of terms in a fourier series, we can measure the quality of this approximation.

$$\int (f(t) - a_0/2 - \sum_1^N a_k \cos(kt) + \sum b_k \sin(kt))^2 = (\sum_{N+1}^{\infty} a_k \cos(kt) + \sum b_k \sin(kt))^2$$

2.4 Fourier Transform

We've talked about approximating periodic functions over the interval 0 to 2π. It's basically the same to approximate arbitrary functions. First, to approximate periodic functions over a longer interval, we just reparameterize to get something periodic from 0 to 2π, use the Fourier series, and then unreparameterize. For an arbitrary interval, we take the limit of this process.

If $f(t)$ is periodic over the interval 0 to $2\pi l$, we take:

$$f(t) = \sum a_k e^{ikt/l}$$

If we take the limit as $l \to \infty$, this is equivalent to using e^{ikt} for any real value of k. So, instead of taking a sum, we must take an integral.

$$f(t) = \int_{-\infty}^{\infty} F(k)e^{ikt}dk$$

This makes sense, because if we have a long enough interval, any two sine waves with different (real, not integer) frequencies will be orthogonal, because eventually they go completely out of phase. So for example, $\sin(t)$ and $\sin(1.5t)$ will be orthogonal over an interval of 0 to 4π, even though they wouldn't be over an interval of 0 to 2pi.

Notice that to take the inner product of complex numbers we must take the complex conjugate of one. So, for example, the inner product of $(1+i)$ with itself is not $1^2 + i^2 = 0$, but is $1^2 - i^2 = 2$. So, we have:

$$F(k) = \int_{-\infty}^{\infty} f(t)e^{-ikt}dt$$

So inverting the fourier transform is almost the same as taking it.