Lighting affects appearance
How do we represent light? (1)

- Ideal distant point source:
  - No cast shadows
  - Light distant
  - Three parameters
  - Example: lab with controlled light
How do we represent light? (2)

- Environment map: $I(\theta,\phi)$
  - Light from all directions
  - Diffuse or point sources
  - Still distant
  - Still no cast shadows.
  - Example: outdoors (sky and sun)
Lambertian + Point Source

\[ \vec{l} = l \cdot \vec{l} \]

- \( \vec{l} \) is direction of light
- \( l \) is intensity of light

\[ i = \max(0, \lambda(\vec{l} \cdot \hat{n})) \]

- \( i \) is radiance
- \( \lambda \) is albedo
- \( \hat{n} \) is surface normal

\( \theta \)
Lambertian, point sources, no shadows. (Shashua, Moses)

- Whiteboard
- Solution linear
- Linear ambiguity in recovering scaled normals
- Lighting not known.
- Recognition by linear combinations.
Linear basis for lighting

λZ

λX

λY
A brief Detour: Fourier Transform, the other linear basis

- Analytic geometry gives a coordinate system for describing geometric objects.
- Fourier transform gives a coordinate system for functions.
Basis

- $P=(x,y)$ means $P = x(1,0)+y(0,1)$
- Similarly:

$$f(\theta) = a_{11} \cos(\theta) + a_{12} \sin(\theta)$$
$$+ a_{21} \cos(2\theta) + a_{22} \sin(2\theta) + \ldots$$

Note, I’m showing non-standard basis, these are from basis using complex functions.
Example

\[ \forall c, \exists a_1, a_2 \text{ such that: } \]
\[ \cos(\theta + c) = a_1 \cos \theta + a_2 \sin \theta \]
Orthonormal Basis

- $\|(1,0)\| = \|(0,1)\| = 1$
- $(1,0).(0,1) = 0$
- Similarly we use normal basis elements eg:
  \[
  \begin{align*}
  \frac{\cos(\theta)}{\|\cos(\theta)\|} & \quad \|\cos(\theta)\| = \sqrt{\int_0^{2\pi} \cos^2 \theta \, d\theta} \\
  \int_0^{2\pi} \cos \theta \sin \theta \, d\theta & = 0
  \end{align*}
  \]
- While, eg:
2D Example
Convolution

\[ f(x) = g * h = \int g(x - x_0) h(x_0) \, dx_0 \]

Imagine that we generate a point in \( f \) by centering \( h \) over the corresponding point in \( g \), then multiplying \( g \) and \( h \) together, and integrating.
Convolution Theorem

\[ f \otimes g = T^{-1} F \ast G \]

- \( F, G \) are transform of \( f, g \)

That is, \( F \) contains coefficients, when we write \( f \) as linear combinations of harmonic basis.
Examples

\[
\cos \theta \otimes \cos \theta = ?
\]

\[
\cos \theta \otimes \cos 2\theta = ?
\]

\[
\cos \theta \otimes f = ?
\]

\[
(\cos \theta + 0.2 \cos 2\theta + 0.1 \cos 3\theta) \otimes f = ?
\]

Low-pass filter removes low frequencies from signal. Hi-pass filter removes high frequencies. Examples?
Shadows

- Attached Shadow
- Cast Shadow
With Shadows: PCA

(Epstein, Hallinan and Yuille; see also Hallinan; Belhumeur and Kriegman)

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Dimension: $5 \pm 2D$
Domain

- Lambertian
- Environment map

\[ \lambda_{\text{max}} (\cos \theta, 0) \]
Lighting to Reflectance: Intuition
(See D’Zmura, ‘91; Ramamoorthi and Hanrahan ‘00)
Spherical Harmonics

- Orthonormal basis, $h_{nm}$, for functions on the sphere.
- $n$’th order harmonics have $2n+1$ components.
- Rotation = phase shift (same $n$, different $m$).
- In space coordinates: polynomials of degree $n$.
- S.H. used for BRDFs (Cabral et al.; Westin et al;). (See also Koenderink and van Doorn.)

\[ h_{nm}(\theta, \phi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_{nm}(\cos \theta)e^{im\phi} \]

\[ P_{nm}(z) = \frac{(1-z^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{dz^{n+m}}(z^2 - 1)^n \]
S.H. analog to convolution theorem

• Funk-Hecke theorem: “Convolution” in function domain is multiplication in spherical harmonic domain.

• $k$ is low-pass filter.
Harmonic Transform of Kernel

\[ k(\theta) = \max(\cos \theta, 0) = \sum_{n=0}^{\infty} k_n h_{n0} \]

\[ k_n = \begin{cases} 
\frac{\sqrt{\pi}}{2} & n = 0 \\
\frac{\sqrt{\pi}}{\sqrt{3}} & n = 1 \\
(-1)^{n+1} \frac{(n - 2)! \sqrt{(2n + 1)\pi}}{2^n \left(\frac{n}{2} - 1\right)! \left(\frac{n}{2} + 1\right)!} & n \geq 2, \text{ even} \\
0 & n \geq 2, \text{ odd}
\end{cases} \]
Amplitudes of Kernel

\[ A_n \]

\( n \)

0.886, 0.591, 0.222, 0.037, 0.014, 0.007
Energy of Lambertian Kernel in low order harmonics

Accumulated Energy

37.5 87.5 99.2 99.81 99.93 99.97
Reflectance Functions Near Low-dimensional Linear Subspace

\[ r = k \ast l = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (K_{nm}L_{nm})h_{nm} \]

\[ \approx \sum_{n=0}^{2} \sum_{m=-n}^{n} (K_{nm}L_{nm})h_{nm} \]

Yields 9D linear subspace.
How accurate is approximation?

Point light source

$$r = k \times l = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (K_{nm} L_{nm}) h_{nm} \approx \sum_{n=0}^{2} \sum_{m=-n}^{n} (K_{nm} L_{nm}) h_{nm}$$

9D space captures 99.2% of energy
How accurate is approximation?

(2)

Worst case.

DC component as big as any other.

1st and 2nd harmonics of light could have zero energy

9D space captures 98% of energy
Forming Harmonic Images

\[ b_{nm}(p) = \lambda r_{nm}(X, Y, Z) \]
Compare this to 3D Subspace

\( \lambda Z \)

\( \lambda X \)

\( \lambda Y \)
Accuracy of Approximation of Images

- Normals present to varying amounts.
- Albedo makes some pixels more important.
- *Worst case approximation* arbitrarily bad.
- “*Average*” case approximation should be good.
Models

Find Pose

Query

Compare

Harmonic Images

Matrix: B

Vector: I