

Shape Spaces

1. Can we define a metric space for shapes?
 - Distance between shapes measures perceptual similarity.
 - Nice properties such as triangle inequality.
2. Unique geodesic paths
 - This provides morphing
3. Set of points vs. curves

Assumptions

- Two sets of 2D points.
- Mostly we assume there exists a correct one-to-one correspondence
- And this correspondence is given.
 - This is very natural in morphometrics, where points are measured and labeled.
 - In vision we must solve for correspondence. Next class we'll look at papers that do this.

Shape Space

- What is shape? Qualities of points that don't depend on translation, rotation or scale.
- So describe points independent of similarity transformation.
 1. Remove translation.
 - Simplest way, translate so point 1 is at origin, then remove point one.
 - More elegant, translate center of mass to origin, remove a point.
 2. Scale so that $\sum ||X_i||^2 = 1$.

Resulting set of points is called *pre-shape*.

Pre because we haven't removed rotation yet.

Notation: , U and X denote sets of normalized points.

Points called X_i and U_i , with coordinates (x_i, y_i) , (u_i, v_i) .

Pre-shape

- If we started with n points, we now have $n-1$ so that:
- $\sum x_i^2 + y_i^2 = 1$.
- So we can think of these coordinates as lying on a unit hypersphere in $2(n-1)$ -dimensional space.

Shape

- If we consider all possible rotations of a set of normalized points, these trace out a closed, 1D curve in pre-shape space.
- Distances between shapes can be thought of as distances between these curves.
 - Notice that to compute distance, without loss of generality we can assume that one set of points (U) does not rotate, since rotating both point sets by the same amount doesn't change distances.

Procrustes Distances

- Full Procrustes Distance. D_F
 - $\min(s, \theta) \|U - sXR(\theta)\|$ That is, we find a scaling and rotation of X that minimizes the euclidean distance to U . ($R(\theta)$ means rotate by θ).
- Partial Procrustes Distance. D_P
 - $\min(\theta) \|U - XR(\theta)\|$. That is, rotate X to minimize the euclidean distance to U .
- Procrustes Distance. ρ
 - Rotate X to minimize the geodesic distance on the sphere from X to U .

Linear Pose Solving

- We can linearly find optimal similarity transformation that matches X to U . (ie., minimize $\sum \|AX_i - U_i\|^2$, where A is a similarity transformation.
 - This is asymmetric between X and U .
- In same way we can linearly compute Full Procrustes Distance.
 - This is symmetric.
 - Leads immediately to other procrustes distances.

Linear Pose: 2D rotation, translation and scale

$$\begin{pmatrix} u_1 & u_2 & \cdot & \cdot & \cdot & u_n \\ v_1 & v_2 & & & & v_n \end{pmatrix} = s \begin{pmatrix} \cos \theta & \sin \theta & t_x \\ -\sin \theta & \cos \theta & t_y \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_n \\ y_1 & y_2 & & & & y_n \\ 1 & 1 & & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b & t_x \\ -b & a & t_y \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_n \\ y_1 & y_2 & & & & y_n \\ 1 & 1 & & & & 1 \end{pmatrix}$$

with $a = s \cos \theta$, $b = s \sin \theta$

- Notice a and b can take on any values.
- Equations linear in a , b , translation.
- Solve exactly with 2 points, or overconstrained system with more.

$$s = \sqrt{a^2 + b^2} \quad \cos \theta = \frac{a}{s}$$

Similarity Matching

- Given point sets X and U , compare by finding similarity transformation A that minimizes $\|AX - U\|$.
 - X = points $X_1, \dots, X_n$. U = points $U_1 \dots U_n$.
 - Find A to minimize sum $\|AX_i - U_i\|^2$
 - This is just a straightforward, linear problem.
 - Taking derivatives with respect to four unknowns of A gives four linear equations in four unknowns.

Issues with this approach

- It is asymmetric.
 - Ok when comparing a model to an image.
 - Not so sensible for comparing two shapes.

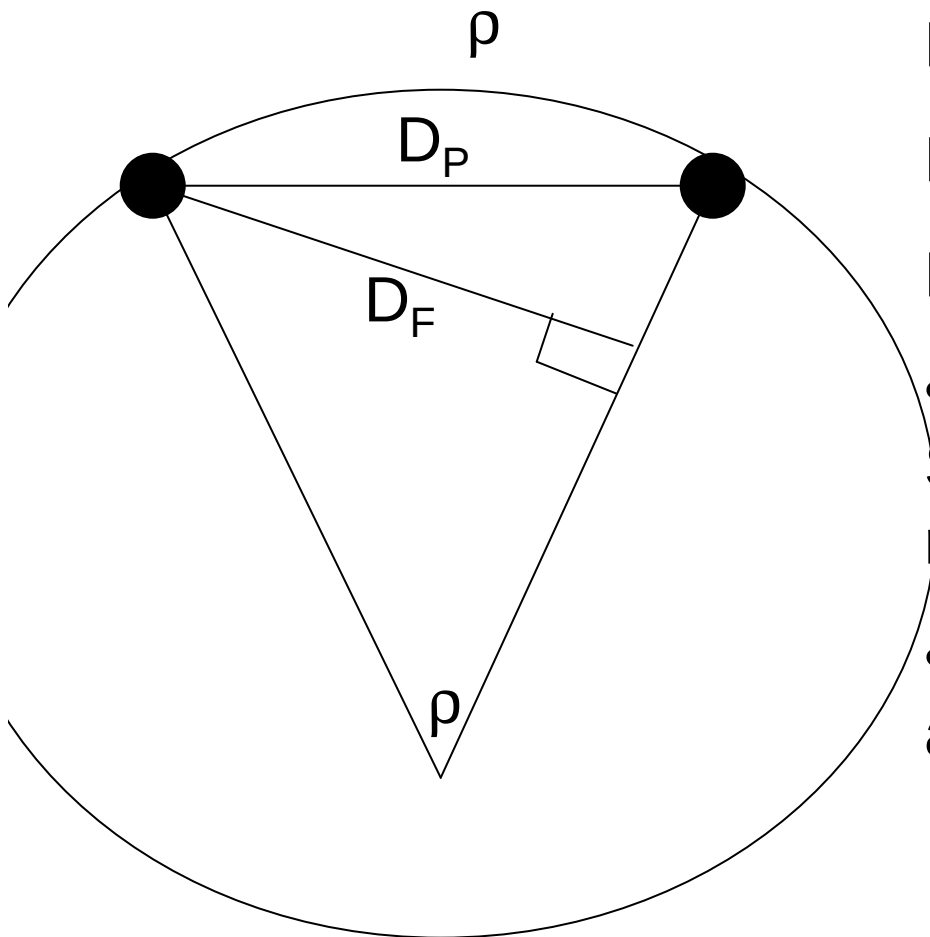
- Note that we now also know how to calculate the Full Procrustes Distance. This is just a least-squares solution to the overconstrained problem:

$$\begin{pmatrix} u_1 & u_2 & \cdot & \cdot & \cdot & u_n \\ v_1 & v_2 & & & & v_n \end{pmatrix} = s \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_n \\ y_1 & y_2 & & & & y_n \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_n \\ y_1 & y_2 & & & & y_n \end{pmatrix}$$

- It is not obvious that Full Procrustes is symmetric.

Given two points on the hypersphere, we can draw the plane containing these points and the origin.



Procrustes Distances is ρ .

$$D_P = 2 \sin (\rho/2)$$

$$D_F = \sin \rho.$$

- These are all monotonic in ρ . So the same choice of rotation minimizes all three.
- D_F is easy to compute, others are easy to compute from D_F .

Why Procrustes Distance?

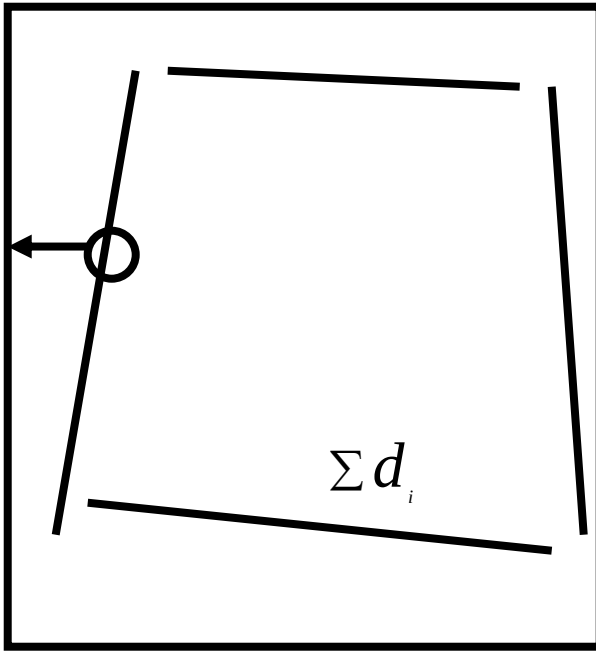
- Procrustes distance is most natural. Our intuition is that given two objects, we can produce a sequence of intermediate objects on a 'straight line' between them, so the distance between the two objects is the sum of the distances between intermediate objects. This requires a geodesic.

Tangent Space

- Can compute a hyperplane tangent to the hypersphere at a point in preshape space.
- Project all points onto that plane.
- All distances Euclidean. Average shape easy to find.
- This is reasonable when all shapes similar.
- In this case, all distances are similar too.
 - Note that when ρ is small, ρ , $2\sin(\rho/2)$, $\sin(\rho)$ are all similar.

Other Point Matching

Approaches: Chamfer Matching

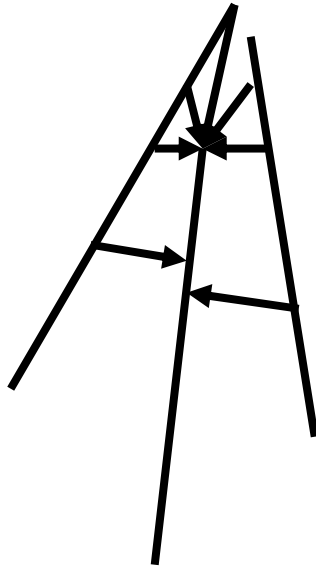


For every edge point in the transformed object, compute the distance to the nearest image edge point. Sum distances.

$$\sum_{i=1}^n \min(\|p_i, q_1\|, \|p_i, q_2\|, \dots, \|p_i, q_m\|)$$

Main Feature:

- Every model point matches an image point.
- An image point can match 0, 1, or more model points.



Then, minimize this distance over pose.

- Example: minimum Chamfer distance over all translations.

$$\min_t \sum_{i=1}^n \min(\| p_i + t, q_1 \|, \| p + t_i, q_2 \|, \dots \| p_i + t, q_m \|)$$

Variations

- Sum a different distance
 - $f(d) = d^2$
 - or *Manhattan distance*.
 - $f(d) = 1$ if $d < \text{threshold}$, 0 otherwise.
 - This is called *bounded error*.
- Use maximum distance instead of sum.
 - This is called: *directed Hausdorff distance*.
 - Use median distance.
- Use other features
 - Corners.
 - Lines. Then position and angles of lines must be similar.
 - Model line may be subset of image line.

Thin-Plate Splines

A function, $f, R^2 \rightarrow R^2$ is a thin-plate spline if:

- **Constraint:** Given corresponding points: $X_1 \dots X_n$ and $U_1 \dots U_n$, $f(X_i) = U_i$.
- **Energy:** f minimizes the following:

$$\iint_{R^2} \left(\left(\frac{\partial^2 f}{\partial x^2} \right)^2 + \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 f}{\partial y^2} \right)^2 \right) dx dy$$

If we think of this as the amount of bending produced by f . Allows arbitrary affine transformation.

- **Solution:** The function f can be computed using straightforward linear algebra. See *Principal Warps: Thin-Plate Splines and the Decomposition of Deformations* by Bookstein, or *Statistical Shape Analysis* by Dryden and Mardia for details.
- **Extension:** Can penalize mismatch of points (using function of $\|U_i - f(X_i)\|$).
- **Results:** Much like D'Arcy Thompson.