Midterm CMSC 828
Instructor: David Jacobs
Assigned, March 28, 2006. Due April 4, 2006 (at start of class).

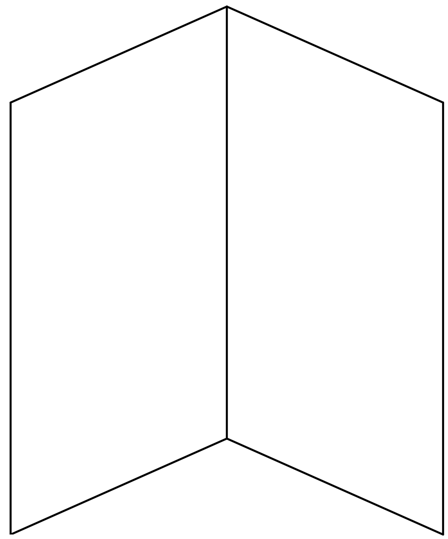
1. **Invariance:**
    a. Suppose we apply an affine transformation to a triangle that has unit area. The affine transformation has a form $Ap+t$ where A is a 2x2 matrix, p is a column vector containing the x and y coordinates of a point, and t is a column vector providing the translation. Prove that the area of the transformed triangle is equal to the determinant of A.
    b. Use this fact to show that the ratio of the area of two triangles is invariant under affine transformations.
    c. Suppose we have two triangles of equal area. We apply a projective transformation to them, which can be expressed in matrix form as:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varepsilon & 1 \end{pmatrix}$$

    What is the maximum possible ratio of the areas of the two triangles after applying this transformation? Give an example of two triangles that achieve this ratio.

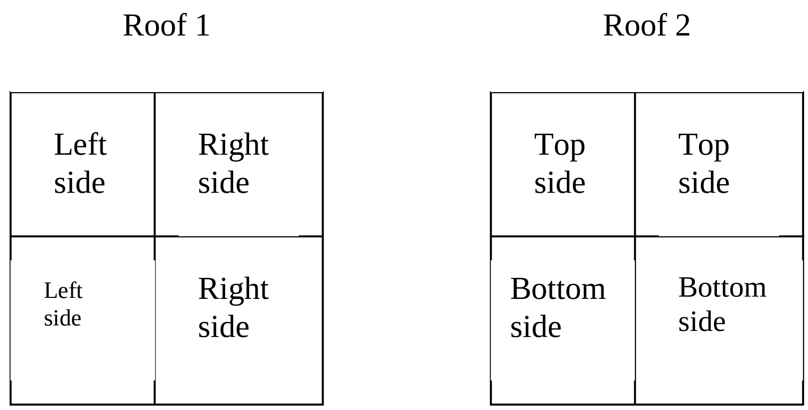
2. **Linear Subspaces**
    a. Suppose we are looking down on a roof from overhead. The roof has two equal sized planar parts, which are at a 90 degree angle from each other. For example, the roof might have vertices at: (0,0,0) (0,2,0), (1,0,1), (1,2,1), (2,0,0), (2,2,0). The picture below shows the roof from a slight angle, so you can see its shape, but imagine looking at it from directly overhead. Further, suppose each side of the roof produces two pixels, so the whole image is 2x2. Now, imagine we generate a set of images of this roof. For each image, we use a single point light source that is infinitely far from the roof. The light sources all have the same intensity, and their directions are uniformly distributed among those directions that face both sides of the roof (that is, neither side of the roof is ever in the dark). The roof itself is made from white, Lambertian material.

    Suppose we perform PCA on this set of images. What will be the most significant component of the images? Note, this problem could be answered analytically, or with a simulation.

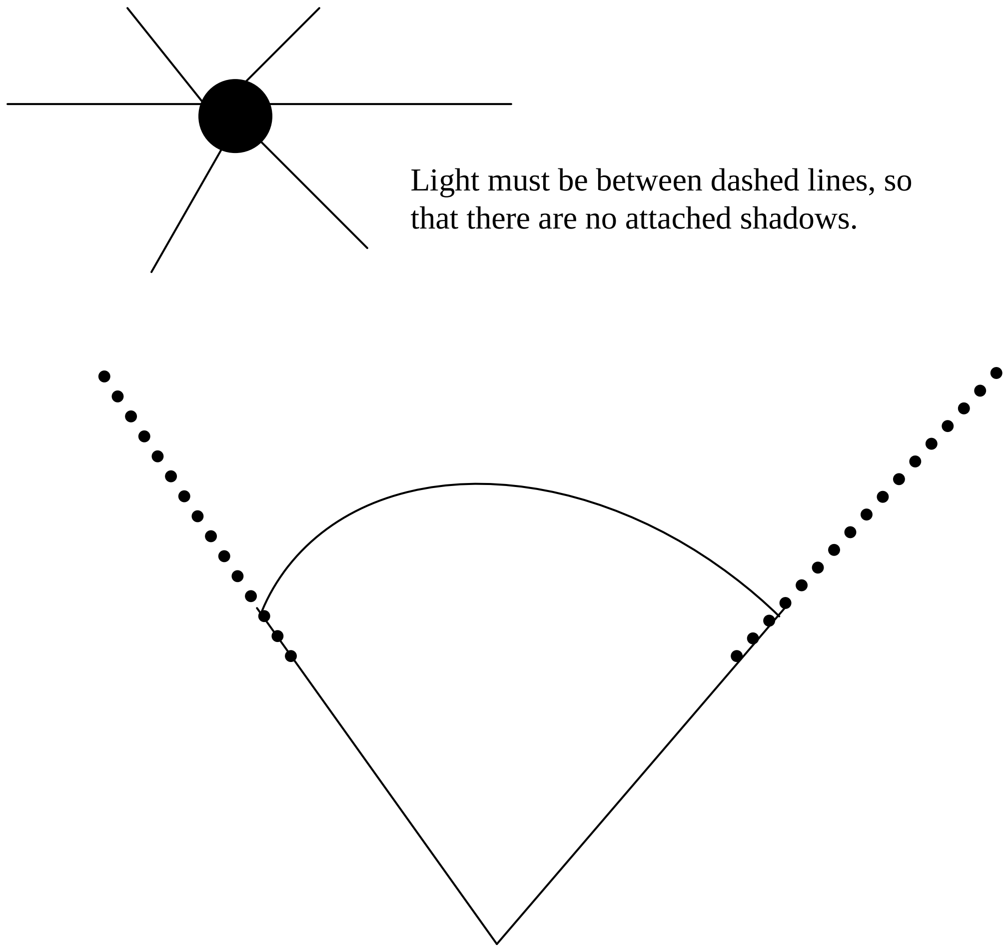


b. Now suppose the lights are not necessarily uniformly distributed, but the distribution of lights is symmetric about the *y*-axis. So, if an image is generated with a light coming from the direction (1,1,2), there will be another light coming from the direction (-1,1,2). What are the possible values for the principle component of the set of images that we produce?

c. Now suppose we have a second object, which is just like the roof described above, but rotated by 90 degrees. So in the new images, the bottom two pixels come from the same side of the roof, whereas in the old pictures the two left-most pixels came from the same side of the roof. We would like to discriminate between these two objects, based on a single image. Suppose we collect images of the two roofs, taken under the lighting conditions described in part (a). We use these images to build the most discriminating one-dimensional subspace using Linear Discrimant Analysis. What 1D subspace will be optimal, according to the criteria of LDA? Explain your answer carefully.



3. **Procrustes Distance:**
    a. Find the procrustes distance between the two point sets: {(2,3), (6,6), (5,7)} and {(7,6), (12,6), (7,1)}. You may account for translation by using the first point as the origin.
    b. Give two point sets, each containing three 2D points, which have the maximum possible procrustes distance. Explain your answer.
    c. **Challenge problem:** Suppose we measure the distance between two triangles by the minimum procrustes distance taken over all permutations of the points in the triangles. For example, the point sets {(0,0), (1,0), (0,1)} and {(0,0), (0,1), (1,0)} do not have a procrustes distance between them, but according to this measure they would represent triangles with zero distance between them. Find the pair of triangles that have the largest possible procrustes distance between them. Prove that your answer is optimal. Partial credit if you can say anything interesting about the answer, even if you can't find it.
4. **Linear Lighting Subspaces**
    a. Suppose we have a scene consisting of a quarter of a cylinder. Consider the set of all images of the cylinder that are each generated by a single, infinitely distant point light source, where the lights are limited to directions in which there are no shadows (every point on the cylinder sees every light source). What is the dimensionality of this set of images?



b. What is the dimensionality if each image can be generated by multiple light sources? Every light source is still constrained so that it does not produce any shadows.

5. Consider a scene consisting of a flat, Lambertian playground with an infinitely thin flag pole. We view the scene from directly above, so that the playground is visible, but the flag pole appears only as a negligible point. Suppose the scene is illuminated by a single infinitely distant light, each of which has an elevation of 45 degrees, and equal intensity. A single directional light illuminates the playground to constant intensity except for a thin, black shadow on it. We suppose that none of these shadows overlap, because the pole is infinitely thin. What do you get when you perform principle component analysis on these images? What does this tell us about the complexity of images produced by shadows, relative to the complexity of images of scenes that do not have cast shadows?
6. Suppose we have three image points located at (8,2), (51,17) and (32,62). In sensing these points, we allow for error in each coordinate of up to three pixels. So, for example, we would match the point at (8,9) to a model point with an *x* coordinate between 5 and 11, and with a *y* coordinate between 6 and 12. Now suppose we have three model points, with locations (0,0), (2,1), (1,3). We want to

match these to the image points allowing for only two degrees of freedom in the model-to-image transformation. The points can be scaled or they can be translated in the *x* direction.

Using the approach of Cass, determine divide the space of transformations into equivalence classes. For each equivalence class of transformations, provide a precise mathematical description of the transformations in that class, and tell which points are matched in that class. How many equivalence classes are there?