Midterm

CMSC 828J, Fall 2009
Assigned, Nov. 2, 2009
Due, Nov. 9, 2009 (in class)

1. **Level Sets**

   (a) **Initial Value Formulation**
   This has the form: $\phi_t + F\|\nabla\phi\| = 0$. Suppose that we have a unit circle, centered at the origin, at time zero, which is expanding at a constant speed of 1.

      i. Give an equation for the initial condition.
      ii. Give an expression for ϕ that is a solution to these equations. Plug it into the initial value equation and show that this is satisfied by the points of a circle centered at the origin that has a radius of 4, at time 3.
      iii. Now suppose instead of solving this analytically, you were to solve it numerically. Explain, with equations, how you could use the initial values to solve for the value of $\phi(1, 0, h)$, where h is a small increment in time. Work these through to provide a numerical value for $\phi(1, 0, h)$. How, if at all, does this differ from the analytic value.

   (b) **Boundary Value Formulation**
   This has the form: $\|\nabla T\|F = 1$. Suppose $T(x, y) = 0$ for all points on a square, centered at the origin, of width 5. That is, $T(x, y) = 0$ for $x = 5$ and $-5 \leq y \leq 5$, for $x = -5$ and $-5 \leq y \leq 5$, for $y = 5$ and $-5 \leq x \leq 5$, and for $y = -5$ and $-5 \leq x \leq 5$.

      i. Using the Fast Marching algorithm, what value would we calculate for $T(6, 3)$?
      ii. What value would we calculate for $T(7, 6)$?

   Where appropriate, give all equations you would need to make these calculations.

2. **Diffusion**
   Let $f(t, 0)$ be an initial distribution of temperature. Suppose $f(t, 0)$ has exactly two local maxima. The temperature evolves over time according to a homogenous, isotropic diffusion. At some subsequent time, can f have more than two local maxima? Either prove that it cannot or give an example demonstrating that it can.

3. **Markov Processes** Suppose Alice and Bob are playing a game. There is a token that starts with Alice. Alice and Bob each have a coin. Whoever has the token, flips their coin. If the coin lands heads, they pass the token to the other player. If the coin lands tails, the player keeps the token. This is repeated a million times. Whoever has the token at the end wins. You get to watch the first move of a few thousand games, and conclude that Alice is using a biased coin, that lands tails 80% of the time.

   (a) If Bob's coin is fair, what is the probability that Alice will win the game?
   (b) Suppose you also get to learn the outcome of a few thousand games, and discover that Bob is winning 60% of the time. Use this information to determine the probability that Bob's coin lands tails when it is flipped.

4. **K-means**

   (a) Suppose we apply K-means clustering to the four points: $(0, -1), (0, 1), (1, 0), (-1, 0)$. We cluster these into two groups based on Euclidean distance, using two random starting points for the cluster centers. How many possible final solutions might we get for different initial conditions?

   (b) Suppose we have n points uniformly distributed on the boundary of a unit circle, where n is a very large number. How many possible final solutions might we get?

5. **Mean Shift**

   Suppose we have three points: $(-1, 0), (0, 0), (1, 0)$. We apply the mean shift algorithm with a starting point of $(-1, 0)$ a Gaussian kernel, and a bandwidth of 1.

   (a) What will happen on the first iteration of Mean Shift?

   (b) If we consider different starting points for the algorithm, how many different points of convergence could we wind up at?

   (c) Give an example of a different bandwidth that would produce a different number of points of convergence.

   (d) Prove that when we use a Gaussian kernel, mean shift is performing gradient ascent on the underlying distribution formed by the points, using Kernel Density Estimation.