Problem 1. Write the factorization process as

\[ L_{n-1} P_{n-1} L_{n-2} P_{n-2} \cdots L_2 P_2 L_1 P_1 A = U. \]

Note that \( L_{n-1} \) is invertible and the inverse has the same form as \( L_{n-1} \), which gives

\[ P_{n-1} L_{n-2} P_{n-2} \cdots L_2 P_2 L_1 P_1 A = L_{n-1}^{-1} U = \hat{L}_{n-1} U. \]

Also, permutation matrices are invertible, and the ones in use here, which interchange two indices, are symmetric and satisfy \( P_k P_k = I \). Thus, we can rewrite the equation above as

\[ P_{n-1} L_{n-2} (P_{n-1} P_{n-1}) P_{n-2} \cdots L_2 P_2 L_1 P_1 A = \hat{L}_{n-1} U. \]

The permutation matrix \( P_k \) interchanges index \( k \) and some index \( k' \) such that \( k + 1 \leq k' \leq n \). It follows that \( P_{n-1} L_{n-2} P_{n-1} \) has the same structure as \( L_{n-2} \), and similarly its inverse also has this structure. Defining \( \hat{L}_{n-2} = P_{n-1} L_{n-2}^{-1} P_{n-1} \), we have

\[ P_{n-1} P_{n-2} \cdots L_2 P_2 L_1 P_1 A = \hat{L}_{n-2} \hat{L}_{n-1} U. \]

Continue in this manner, choosing \( \hat{L}_k = P_{n-1} \cdots P_{k+1} L_k^{-1} P_{k+1} \cdots P_{n-1} \), which leads to

\[ P_{n-1} P_{n-2} \cdots P_2 P_1 A = \hat{L}_1 \cdots \hat{L}_{n-2} \hat{L}_{n-1} U. \]

Problem 2.

a. \( Q \) is symmetric so that \( QQ^T = Q^T Q \). Expansion of the product gives

\[ M \left( I - \frac{2}{u^T u} uu^T \right) \left( I - \frac{2}{u^T u} uu^T \right) = I - \frac{4}{u^T u} uu^T + \frac{4}{(u^T u)^2} uu^T uu^T = I. \]

b. \( Qu = u - \frac{2}{u^T u} uu^T u = -u \). If \( u^T v = 0 \), then it is clear that \( Qv = v \).

c. For the case of \( \mathbb{R}^3 \), let

\[ x = \alpha_1 u + \alpha_2 u_2 + \alpha_3 u_3. \]

By definition, \( u^T u_j = 0 \) for \( j = 2 \) and \( 3 \), so that \( u^T x = \alpha_1 \). It then follows that

\[ x - 2 uu^T x = \alpha_1 u + \alpha_2 u_2 + \alpha_3 u_3 - 2 \alpha_1 u \]

\[ = -\alpha_1 u + \alpha_2 u_2 + \alpha_3 u_3. \]

That is, the Householder transformation \( Qx \) reflects \( x \) through the plane spanned by \( u_2 \) and \( u_3 \) along the direction of \( u \). A depiction of the reflection is below. The analysis for \( \mathbb{R}^n \) is identical, where the reflection is through the hyperplane spanned by \( \{ u_2, \ldots, u_n \} \).
d. Because \( \|x\| = \|y\| \), we have
\[
(x - y, x + y) = (x, x) + (x, y) - (y, x) - (y, y) = \|x\|^2 - \|y\|^2 = 0.
\]
Therefore, with the Householder matrix \( Q \) defined from \( u = x - y \), the result from part (b) implies
\[
Q(x - y) = y - x, \quad Q(x + y) = x + y.
\]
But \( x = \frac{1}{2}(x + y + x - y) \), which leads to
\[
Qx = \frac{1}{2} (Q(x + y) + Q(x - y)) = \frac{1}{2} (x + y + y - x) = y.
\]

e. It is clear that if \( Q_1 \) is defined using the choices of \( x \) and \( y \) proposed, then \( Q_1A \) has \( \|a_1\| e_1 \) in its first column. For the second step, the required transformation has the form
\[
Q_2 = \begin{pmatrix}
1 & 0 \\
0 & \tilde{Q}_2
\end{pmatrix}
\]
where \( \tilde{Q}_2 \) is a Householder matrix of order \( n - 1 \) that produces zeros below the diagonal in the second column of \( Q_1A \). The computation proceeds in this way: at the \( k \)th step,
\[
Q_k = \begin{pmatrix}
I & 0 \\
0 & \tilde{Q}_k
\end{pmatrix}
\]
where the identity is of order \( k - 1 \). The result is \( Q_{n-1} \cdots Q_2Q_1A = R \) upper triangular, or
\[
A = QR \quad \text{with} \quad Q = (Q_{n-1} \cdots Q_2Q_1)^T.
\]

**Problem 3.**

a. The argument is by induction. The definition of \( \delta_1 \) ensures that \( v^{(2)} \) is orthogonal to \( v^{(1)} \), which is the base case for the induction. Now assume \( (v^{(i)}, v^{(j)}) = 0 \) for \( i, j \leq k < N, i \neq j \).
Consider
\[
(v^{(k+1)}, v^{(j)}) = \frac{1}{\gamma_{k+1}^{(j)}} \left[ (Av^{(k)}, v^{(j)}) - \delta_{k}(v^{(k)}, v^{(j)}) - \gamma_{k}(v^{(k-1)}, v^{(j)}) \right].
\]
In the case \( j = k \), the last term on the right is zero by the induction hypothesis, and the rest of the expression is zero by the definition of \( \delta_{j} \). When \( j < k \), we use the symmetry of \( A \) and the recurrence at step \( j \) to write the first expression inside the brackets as
\[
(Av^{(k)}, v^{(j)}) = (v^{(k)}, Av^{(j)}) = \gamma_{j+1}(v^{(k)}, v^{(j+1)}) + \delta_{j}(v^{(k)}, v^{(j)}) + \gamma_{j}(v^{(k)}, v^{(j-1)}).
\]
Then if \( j < k - 1 \), all the inner products are 0 by the induction hypothesis. It remains to consider \( j = k - 1 \); in this case the only terms present are
\[
\gamma_{k}(v^{(k)}, v^{(k)}) - \gamma_{k}(v^{(k-1)}, v^{(k-1)}) = \gamma_{k} - \gamma_{k} = 0.
\]
b. The three-term recurrence can be written in matrix form as
\[
AV_k = V_kT_k + \gamma_{k+1} \begin{bmatrix} 0, \ldots, 0, v^{(k+1)} \end{bmatrix}
\]
where $\mathbf{0}$ here refers to a vector of length $N$ consisting of all zeros. Premultiplying this relation by $V_k^T$ and using the fact that all the vectors $v^{(j)}$ are orthogonal leads to $V_k^T A V_k = T_k$.

c. Showing that the residual is orthogonal to the columns of $W_k$ is equivalent to showing $W_k^T (A W_k - W_k \Lambda_k) = 0$. Noting that $V_k^T V_k = I$ and $U_k^T U_k = I$, we have

$$W_k^T (A W_k - W_k \Lambda_k) = U_k^T V_k^T (AV_k U_k - V_k U_k \Lambda_k) = U_k^T V_k^T A V_k U_k - U_k^T V_k^T V_k U_k \Lambda_k = U_k^T T_k U_k - \Lambda_k.$$ 

The eigendecomposition of $T_k = U_k \Lambda U_k^T$ is equivalent to $\Lambda_k = U_k^T T_k U_k$. Therefore, $W_k^T (A W_k - W_k \Lambda_k) = 0$.

**Problem 4.**

a. See `lanczos.m`. It takes 44 iterations to produce the top 5 eigenvalues: 4.008, 4.008, 4.008, 2.220, 2.220. The lanczos algorithm has produced the two maximum eigenvalues produced by `eig`. Note that the residuals of the smaller eigenvalues are much larger than those of the larger eigenvalues.

b. To compute the smallest eigenvalues, we can perform lanczos method on $A^{-1}$. This algorithm produces the five smallest eigenvalues 0.0050, 0.0050, 0.0050, 0.0050, 0.1624. Which takes 29 iterations to converge. These are the two smallest eigenvalues produced by `eig`.

c. The 5 largest eigenvalues are produced after 30 iterations and are 1.423, 1.425, 1.618, 2.220, 4.008. The 5 smallest eigenvalues are produced after 29 iterations and are 0.0050, 0.1624, 0.2638, 0.3352, 0.3659. These are in agreement with the results of `eig`.

Note that reorthogonalization is needed to obtain a correct representation of the eigenvalues sought. The process given in the assignment is too expensive for practical computation, but “partial reorthogonalization” methods have been devised that are cheaper and equally effective.