Problem 1. Let $A$ be a square nonsingular matrix. Suppose we seek the solution to a linear system of equations $Ax = b$, but instead because of errors in floating point computations, we compute an approximate solution $\hat{x}$ that satisfies a perturbed problem $A\hat{x} = b + \delta b$.

a. Define the condition number of $A$, $\kappa(A)$.

b. Show that the computed vector $\hat{x}$ satisfies

$$\frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}$$

c. State conditions on $\delta b$ and $\kappa(A)$ that would give you confidence in the accuracy of the computed solution $\hat{x}$.

Problem 2. Let

$$A = \begin{bmatrix} a_1, a_2, \ldots, a_n \end{bmatrix}$$

be a matrix of full rank, where for each $j$, $a_j$ is a column vector of length $m$ with $m > n$. Consider the following block of pseudocode:

```pseudocode
for k = 1 to n
    \hat{q}_k \leftarrow a_k
    for i = 1 to k - 1
        r_{ik} \leftarrow a_k^T q_i
        \hat{q}_k \leftarrow \hat{q}_k - q_i r_{ik}
    end
    r_{kk} \leftarrow \sqrt{\hat{q}_k^T \hat{q}_k}
    q_k \leftarrow \hat{q}_k / r_{kk}
end
```

a. Describe in words what this program is doing, and state a mathematical relation about the result of running it.

b. Suppose the definition of $r_{ik}$ above is replaced by $r_{ik} \leftarrow \hat{q}_k^T q_i$. If the computations are done in real (exact) arithmetic, will this change have any impact on the result?

c. Why might the expression from part (b) be preferable to that from part (a) when floating point arithmetic is used?

Comment: For this part, it will suffice to examine what happens in the case $k = 3$. 1
Problem 3. Suppose we want to approximate
\[ I(f) = \int_{\Omega} f(x) dx \]
on some region \( \Omega \subset \mathbb{R}^d \).

a. State an interpretation of \( I(f) \) as an expected value of some random variable.

b. Define the Monte Carlo method for estimating \( I(f) \) and identify a scenario in which you expect it to be effective. No proof or details are required here, just a brief statement.

c. Suppose we have a function \( \rho(x) > 0 \) on \( \Omega \) that satisfies \( \int_{\Omega} \rho(x) dx = 1 \). Then
\[ I(f) = \int_{\Omega} \left( \frac{f(x)}{\rho(x)} \right) \rho(x) dx. \]
State an interpretation of \( I(f) \) expressed this way as an expected value.

d. The method of \textbf{importance sampling} chooses samples in a special way and estimates \( I(f) \) in a manner analogous to Monte Carlo simulation. Specify this computational strategy.

e. Identify a circumstance in which this approach is likely to be more accurate than ordinary Monte Carlo integration. Again, no detailed proof or analysis is needed here.