Scientific Computing I
CMSC / AMSC 660, Fall 2012
Midterm
Solutions

Problem 1.
a. A simple method for solving IVP explicitly is the forward Euler method, which is defined by
\[ y_{k+1} = y_k + (t_{k+1} - t_k) f(t_k, y_k). \]
b. Truncation error is the error \( y_{k+1} - y(t_{k+1}) \) produced by taking a single step of the numerical method assuming the solution at the current step is exact, i.e., \( y_k = y(t_k) \). The truncation error for Forward Euler is \( O(h^2) \) where \( h \) is the length of the time step.
c. An example of an implicit method is backward Euler method,
\[ y_{k+1} = y_k + (t_{k+1} - t_k) f(t_{k+1}, y_{k+1}). \]
The downside of such a method is that it requires solving an equation (or system of equations) to find \( y_{k+1} \). It may be preferable to use implicit methods over explicit methods when small time steps are required for the explicit method to be stable. In such cases, it is often more efficient to use implicit methods that allow larger time steps even with the overhead of equation solution.
d. To ensure the accuracy of the computation, the error estimate can be used to check if the local error at a given step is below some tolerance. If not, the step size can be decreased and the discrete solution recomputed. To enhance the efficiency of the computation, if the error estimate is much smaller than the tolerance, the step size can be increased for the next time step.

Problem 2.
a. \((A^T A)^T = A^T (A^T)^T = A^T A\). So \(A^T A\) is symmetric.
b. A matrix \(B\) is positive-definite if for all vectors \(x\) such that \(x \neq 0\), \(x^T B x > 0\). For \(A^T A\), \(x^T A^T A x = \|Ax\|^2 \geq 0\) for all \(x\). Since \(A\) is full rank, \(Ax = 0\) if and only if \(x = 0\). Therefore, \(x^T A^T A x > 0\) for all \(x \neq 0\), and \(A^T A\) is positive-definite.
c. The linear least squares problem with \(m = 3\), \(n = 2\) case is to find the projection of \(y\) onto the plane formed by the column span of \(A\). A picture of the geometry is below.

Problem 3.
i. Number of multiplications: \(n\). Number of reads: \(2n\). Number of writes: 1.
\[ \hat{\alpha} = \frac{\alpha n + \beta (2n + 1)}{n} = \alpha + \beta (2 + 1/n). \]
ii. Number of multiplications: $n^2$. Number of reads: $n^2 + n$. Number of writes: $n$.

$$\hat{\alpha} = \frac{\alpha n^2 + \beta(n^2 + 2n)}{n^2} = \alpha + \beta(1 + 2/n).$$


$$\hat{\alpha} = \frac{\alpha n^3 + \beta(2n^2 + n^2)}{n^3} = \alpha + \beta(3/n).$$

The matrix-matrix product makes the most effective use of computer resources because the rows of $A$ and the columns of $B$ are used for more than one multiplication. We can also see this because $\beta$ is multiplied by a smaller coefficient than the other two products. Even if $\beta$ is large, the impact of this is mitigated by its smaller coefficient.