Advanced Numerical Linear Algebra Homework 2
CMSC 760 / AMSC 600 Due October 13, 2009

Problem 1. Consider the linear system of equations $Ax = b$ where A is nonsingular. We have seen that when A is nonsymmetric, if the GMRES algorithm is used to solve this problem, it depends on a computation that requires k vectors at step k. Show that when A is symmetric, this computation can be done using a recurrence of fixed length at each step.

Problem 2. When the damped Jacobi algorithm is used as a smoother for multigrid applied to the one-dimensional diffusion parameter, the damping parameter $\omega = 2/3$ is optimal in the sense that the damping factor is the same for the highest frequency mode and the "middle frequency" mode, i.e., the eigenvector of lowest frequency among the set of high frequency modes. The goal here is to identify the analogous parameter for the two-dimensional problem, i.e., the Poisson equation

$$-(u_{xx} + u_{yy}) = f$$

on the unit square $\Omega = [0, 1] \times [0, 1]$, subject to Dirichlet boundary conditions $u = 0$ on $\partial\Omega$.

a. Let A come from the finite difference discretization of this problem using a uniform $n \times n$ grid of interior grid points. Let $h = \frac{1}{n+1}$. Show that up to scaling, A is a block tridiagonal matrix of the form

$$tridiag[-I, T, -I]$$

where I is the identity matrix of order n and T is the tridiagonal matrix of order n given by

$$T = tridiag[-1, 4, -1].$$

Show that the eigenvectors of A are given by the product of the eigenvectors of the one-dimensional problem, that is

$$v^{(j,k)} = \sin j\pi x_j \, \sin k\pi y_k \tag{1}$$

where $x_j = jh$, $y_k = kh$.

b. We say that the oscillatory modes of this problem are those in which at least one of the indices j or k of (1) is greater than or equal to $n/2$. (Thus, approximately 3/4 of the eigenvectors are oscillatory.) Find the optimal damping parameter for the damped Jacobi smoother, i.e., the parameter such that the smoothing factor associated with all the oscillatory modes is as small as possible.

Problem 3. Let $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ be generated by the Arnoldi process for a matrix A, with square Hessenberg matrix H_k.

a. Suppose $v_{k+1} = 0$. Show that in this case, the eigenvalues of H_k are eigenvalues of A, and identify the associated eigenvectors.

b. More generally, if $v_{k+1} \neq 0$, μ is an eigenvalue of H_k and an *estimate* for an eigenvector of A is generated in a manner anagous to what is done for part (a), show that the residual $Av - \mu v$ is orthogonal to the Krylov subspace $K_k(A, v_1)$.