Advanced Numerical Linear Algebra Homework 3
CMSC 760 / AMSC 600 Due November 5, 2009

Problem 1. Let A be a matrix of order n, and let $u_1, u_2, \ldots, u_k$ be eigenvectors of A, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Let $q = c_1 u_1 + \cdots + c_k u_k$.

a. Show that the Krylov subspaces $\mathcal{K}_m(A, q)$ lie in $span\{u_1, \ldots, u_k\}$ for all m.

b. Show that the Arnoldi process starting with q will terminate in $m \leq k$ steps with $span\{v_1, \ldots, v_m\}$ an invariant subpace of A contained in $span\{u_1, \ldots, u_k\}$.

c. Show that if the k eigenvalues above are distinct and $c_i \neq 0$ for $i = 1, \ldots, k$, then the Arnoldi process terminates in exactly k steps.

d. Suppose some of the eigenvalues are equal. Show that if there are $s < k$ distinct eigenvalues, then q can be written as a linear combination of s or fewer eigenvectors of A, and the Arnoldi process will terminate in s steps or fewer.

Problem 2. Let

$$AV_m = V_m H_m + v_{m+1} h_{m+1,m} e_m^T,$$

and let p be a polynomial of degree $j < m$. Show that

$$p(A)V_m = V_m p(H_m) + E_j$$

where $E_j \in \mathbb{C}^{n \times m}$ is identically zero except in the last j columns.

Problem 3. Let V_m, H_m and $h_{m+1,m}$ be generated by the Arnoldi process. Let μ be an eigenvalue of H_m with eigenvector w, normalized so that $\|w\|_2 = 1$. Let v be the Ritz vector derived from w, also normalized so that $\|v\|_2 = 1$. Show that the eigenvalue residual satisfies

$$\|Av - \mu v\|_2 = |h_{m+1,m}|\,|x_m|.$$

Problem 4. Consider the linear system of equations $Ax = b$ where A is symmetric. Let V_k and T_k is derived from the Lanczos process for A i.e.,

$$AV_k = V_k T_k + \delta_{k+1} v_{k+1} e_k^T$$

where V_k contains orthonormal columns and T_k is tridiagonal.

a. Show that if A is positive-definite, then so is T_k.

b. Suppose we seek $x_k = V_k a_k$ is such that $V_k^T(b - Ax_k) = 0$. Describe an efficent computational procedure for finding x_k.

c. What can go wrong if A is not positive-definite? Can you think of a way to fix it?

d. Returning to positive-definite case, derive a bound on the error $\|x - x_k\|_A$. You may use whatever you know about other solvers.

Problem 5. Write your own version of the subspace iteration algorithm and use it to compute some eigenvalues of the matrix X given in the Matlab mat-file `hw3*.mat`. Test this algorithm with various dimensions of subspaces, including 5, 10, 25 and 35, and describe what happens.