Advanced Numerical Linear Algebra Homework 1
AMSC/CMSC 763 Due September 19, 2017

1. Let $Ax = b$ be a linear system of equations where A is symmetric and positive-definite. Let x_k be the kth iterate generated by the conjugate gradient method (CG). Show that if $x_k \neq x$, then the vectors generated by CG satisfy

$$\begin{array}{rll} \text{(i)} & \langle r_k, p_j \rangle = \langle r_k, r_j \rangle = 0, & j < k, \\ \text{(ii)} & \langle Ap_k, p_j \rangle = 0, & j < k, \\ \text{(iii)} & \text{span}\{r_0, r_1, \ldots, r_{k-1}\} = \text{span}\{p_0, p_1, \ldots, p_{k-1}\} & \\ & \quad = \mathcal{K}(A, r_0) \equiv \text{span}\{r_0, Ar_0, \ldots, A^{k-1}r_0\}. & \end{array}$$

Hint: Prove (i) and (ii) simultaneously by induction on k, and use a dimensionality argument for (iii).

2. We defined the Chebyshev polynomial in class as $\tau_k(z) = \cos(k \arccos z)$ when z is a real number with $|z| \leq 1$.

a. Show that the roots of $\tau_k(z)$ are $\left\{ \cos\left(\frac{(2j-1)\pi}{2k}\right) \,\middle|\, j = 1, 2, \ldots, k \right\}$.

b. For $k \geq 1$, let $\tilde{\tau}_k(z) = \left(\frac{1}{2^{k-1}}\right) \tau_k(z)$, so that $\tilde{\tau}_k(z)$ is a polynomial of degree k with leading coefficient equal to 1. Prove that

$$\max_{z \in [-1,1]} |\tau_k(z)| \leq \max_{z \in [-1,1]} |T_k(z)|$$

where T_k is any other polynomial of degree k with leading coefficient 1.

c. For $|z| > 1$, let $\tau_k(z) = \cosh(k \operatorname{arccosh} z)$. Show that in this case, $\tau_k(z)$ satisfies the same recurrence derived in class,

$$\tau_{k+1}(z) = 2z\tau_k(z) - \tau_{k-1}(z).$$

d. Prove that

$$\tau_k(t) = \frac{1}{2} \left[(t + \sqrt{t^2 - 1})^k + (t - \sqrt{t^2 - 1})^k \right].$$

e. Use the result of part (d) to show that

$$\tau_k \left(\frac{b + a}{b - a} \right) > \frac{1}{2} \left(\frac{\sqrt{b/a} + 1}{\sqrt{b/a} - 1} \right)^k.$$

3. Let $Ax = b$ be as in Problem 1. Starting from an arbitrary initial iterate x_0, the steepest descent method generates a sequence of iterates $x_1, x_2, \ldots$ by the computation

$$x_{k+1} = x_k + \alpha_k r_k\,,$$

where r_k is the residual $b - Ax_k$ and α_k is a scalar chosen so that the norm $\|x - x_{k+1}\|_A$ is minimal.

a. Explain the name "steepest descent method."

b. Show that the error $e_k = x - x_k$ satisfies

$$\|e_k\|_A \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^k \|e_0\|_A \,.$$

where $\kappa = \Lambda/\lambda$ is the *condition number* of A, that is, the ratio of the largest eigenvalue of A to its smallest eigenvalue.

4a. Write your own version of the conjugate gradient algorithm and use it to solve the problem $Ax = b$ where A and b are given in the MATLAB mat-file `hw1.mat`. The program should take as input the matrix A, right hand side b, a maximum number of iterations, and a tolerance τ. For output, it should produce the computed solution, a vector containing all the residual norms, and a flag indicating whether the stopping tolerance

$$\|r_k\|_2 \leq \tau \|b\|_2$$

is satisfied. Use this algorithm to compute an approximate solution x_k for which the Euclidean norm of the residual satisfies

$$\|r_k\|_2 \leq 10^{-8} \|b\|_2,$$

where $x_0 = 0$. Plot the residual norm (on a logorithmic scale) against the iteration counts, and identify the number if iterations required.

b. Modify the program so that at each step, the matrix

$$S_k = tridiag \left[-\frac{1}{\alpha_{j-1}}, \frac{1}{\alpha_j} + \frac{\beta_{j-1}}{\alpha_{j-1}}, -\frac{\beta_j}{\alpha_j} \right]$$

is also constructed. (Note that S_k contains S_{k-1} as a principal minor.) As the iteration proceeds, compute the eigenvalues of S_k and compare them to those of A. You can use the Matlab function `eig` to compute the eigenvalues in each case. What do you observe?