Advanced Numerical Linear Algebra Homework 3
AMSC/CMSC 763 Due October 29, 2019

Problem 1. The Chebyshev polynomials are defined for real numbers z with $|z| \leq 1$ as

$$\tau_k(z) = \cos(k \arccos z).$$

a. Show that the roots of $\tau_k(z)$ are $\left\{ \cos\left(\frac{(2j-1)\pi}{2k}\right) \,\middle|\, j = 1, 2, \ldots, k \right\}$.

b. For $k \geq 1$, let $\tilde{\tau}_k(z) = \left(\frac{1}{2^{k-1}}\right) \tau_k(z)$, so that $\tilde{\tau}_k(z)$ is a polynomial of degree k with leading coefficient equal to 1. Prove that

$$\max_{z\in[-1,1]} |\tau_k(z)| \leq \max_{z\in[-1,1]} |T_k(z)|$$

where T_k is any other polynomial of degree k with leading coefficient 1.

c. For $|z| > 1$, let $\tau_k(z) = \cosh(k \operatorname{arccosh} z)$. Show that in this case, $\tau_k(z)$ satisfies the recurrence

$$\tau_{k+1}(z) = 2z\tau_k(z) - \tau_{k-1}(z).$$

d. Prove that

$$\tau_k(t) = \frac{1}{2}\left[(t + \sqrt{t^2-1})^k + (t - \sqrt{t^2-1})^k\right].$$

e. Use the result of part (d) to show that

$$\tau_k\left(\frac{b+a}{b-a}\right) > \frac{1}{2}\left(\frac{\sqrt{b/a}+1}{\sqrt{b/a}-1}\right)^k.$$

Problem 2.
Consider a matrix A that is skew-Hermitian, that is $A^* = -A$.

(i) Show that the eigenvalues of A are purely imaginary. What additional property do they satisfy in the particular case when A is real skew-symmetric? [Hint: eigenvalues of real matrices come in complex conjugate pairs.] What can you say of a real skew-symmetric matrix of *odd* dimension n?

(ii) Assume that Arnoldi?s procedure is applied to A starting with some arbitrary vector v_1. Show that the algorithm will produce scalars h_{ij} such that

$h_{ij} = 0$ for $i < j - 1$
$Re(h_{ij}) = 0$, $j = 1, 2, \ldots, m$
$h_{j,j+1} = -h_{j+1,j}$, $j = 1, 2, \ldots, m$

(iii) From the previous result show that in the particular case where A is real skew-symmetric and v_1 is real, the Arnoldi vectors $\{v_j\}$ satisfy a two-term recurrence of the form

$$\gamma_{j+1}v_{j+1} = Av_j + \gamma_j v_{j-1}.$$

(iv) Show that the approximate eigenvalues of A obtained from the Arnoldi process are also purely imaginary.

P4oblem 3. Let

$$AV_m = V_m H_m + v_{m+1} h_{m+1,m} e_m^T,$$

and let p be a polynomial of degree $j < m$. Show that

$$p(A)V_m = V_m p(H_m) + E_j$$

where $E_j \in \mathbb{C}^{n \times m}$ is identically zero except in the last j columns.

Problem 4. Write your own version of Arnoldi's method for computing the eigenvalues of a general matrix and explore its performance for computing the eigenvalues of the matrix A given in the Matlab mat-file `matrix.mat`. Test this algorithm with Krylov spaces of various dimensions, including 5, 10, 25 and 35, and describe what happens.

You should use the Matlab function `eig` to compute the eigenvalues of the Hessenberg matrix that is constructed by the algorithm. You may also feel free to use `eig` to look at all the eigenvalues of A. You might also find it interesting to use Matlab's function `eigs` to compute just some of the eigenvalues of A and see how well it does in comparison to your code.

As always, you are also free to use PYTHON instead of MATLAB, by importing the matrix. If you do that, state briefly how you compute the eigenvalues of the Hessenberg matrix and of A.