FINAL FOR CMSC 452
SOLUTIONS

1. (0 points) What is your name?

2. (5 points) Give an example of a language L over the alphabet $\{a, b, c\}$ such that BOTH of the following hold:

   - Every string in L has all three letters in it, and
   - L is context free but not regular.

   No proof required.

   SOLUTION TO PROBLEM 2

   $\{a^n b^n c^n \mid n \in \mathsf{N}\}$

3. (20 points) For this problem you can ASSUME $P \neq NP$. $M_1, M_2, \ldots$ is a standard list of Turing Machines. For each of the following languages state one of the following. NO PROOF REQUIRED.

   - REGULAR
   - CFL but not REGULAR
   - P but not CFL
   - NP but not P.
   - DECIDABLE but not in NP.
   - R.E. BUT NOT DECIDABLE.
   - NOT R.E.

   2 points for every RIGHT answer. 0 points if you leave it blank. -2 points for every WRONG answer. (However, if this results in a negative score then you will just get 0.)

   (a) $A = \{a^n b^n c^m \, : \, n, m \geq 1\}$
   Context Free but not regular. Note that this also answers problem 2.

   (b) $B = \{k \, : \,$ there is an $O(2^{n^{1/k}})$ algorithm for SAT $\}$
   ANSWER: Regular. If $k \notin B$ then all $k' > k$ are not in B. Hence B is either N or some finite set. Hence B is regular.

   (c) $C = \{e \, : \, (\forall x)[M_e$ halts on all of the primes $]\}$
   ANSWER: NOT r.e. One can show $\overline{HALT} \leq_m C$ easily.

   (d) $D = \{a^p b^q c^r \, : \, \min\{p, q, r\} \geq 1000\}$
   ANSWER: REG. Keep track of how many a's b's and c's.

(e) $E = \{a^p b^q c^r \ : \ \max\{p, q, r\} \ge 1000\}$

ANSWER: REG. Keep track of how many a's, b's and c's.

(f) $F = \{a^{4n+1} \ : \ n \in HALT\}$. (For example, if $2 \in HALT$ then $a^9 = aaaaaaaaa \in F$.)

ANSWER: R.E. but not DEC. R.E. the same way that HALT is. Undecidable since $HALT \le_m F$ via $n \in HALT$ iff $a^{4n+1} \in F$.

(g) $G = SUBSEQ(\{a^n b^m \ : \ M_n(m) \downarrow\})$

ANSWER: REG. Its just $a^* b^*$.

(h) $\{x \ : \ x$ is a zip code in America $\}$

ANSWER: finite therefore regular

(i) $I = \{a^n b^n c^n d^n \ : \ n \in \mathsf{N}\}$

ANSWER: P but not CFL.

(j) $J = \{\phi \ : \ \phi$ is a formula in DNF form and ϕ is satisfiable $\}$.

ANSWER: P but not CFL.

4. (15 points) Let

$$A = \{x : \text{ the number of inputs } M_x \text{ halts on is between 10 and 20 }\}.$$

Show that $HALT \le_m A$.

SOLUTION TO PROBLEM 4

(a) Input(x)

(b) Create a machine which does the following.

i. Input(y)

ii. If $y = \{1, 2, \ldots, 15\}$ then run $M_x(x)$. Otherwise DIVERGE.

(c) Output the index of this machine.

If $x \in HALT$ then M_y will halt on $1, 2, \ldots, 15$. Hence $y \in A$.

If $x \notin HALT$ then M_y will halt on 0 inputs. Hence $y \notin A$.

5. (15 points) Recall that in class we had the notation that a finite set was represented by a string of bits (1 for IN, 0 for OUT). For example 11001 is the set $\{0, 1, 4\}$. Also recall that we can represent pairs of sets using the alphabet $\{00, 01, 10, 11\}$ (written up and down). For example

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}$$

represents the pair of sets $\{0, 1\}, \{1, 3\}$.

Using this notation draw a DFA for the language

$$L = \{(A, B) : (\exists x)[x \in A \wedge x \notin B] \wedge (\exists y)[y \in B \wedge y \notin A]\}.$$

Note that all inputs are valid.

SOLUTION TO PROBLEM 5 OMITTED.

6. (15 points) Give the algorithm that does the following: Input is a DFA M. Output is a REG EXPRESSION α such that the language *recognized* by M is the language *generated* by α.

SOLUTION TO PROBLEM 6. This is the $R(i, j, k)$ dynamic program thing.

7. (15 points) We define 3-COL and NOT-10-COL below:

$$3\text{-}COL = \{G : G \text{ is 3-colorable}\}.$$

$$NOT\text{-}10\text{-}COL = \{G : \text{ is NOT 10-colorable}\}.$$

Assume that 3-COL is NP-complete.

Show that if NOT-10-COL is in P then $P = NP$.

SOLUTION TO PROBLEM 7.

Let

$$10\text{-}COL = \{G \text{ is 10-colorable}\}.$$

We show that 3-$COL \le_m^p$ 10-COL. We define a function f such that, for all graphs G,

$$G \in 3\text{-}COL \text{ iff } f(G) \in 10\text{-}COL.$$

Given G add 7 new vertices and connect them to each other and to all of the vertices in G. This is $f(G)$. Call this new graph G'.

If G is 3-colorable then G' is 10-colorable by just using the 3-coloring for the G-part, and then color the additional 7 with 7 new colors.

If G' is 10-colorable then look at the coloring of the G-part. It can only use 3 colors since the 7 additional vertices used 7 diff colors that cannot be used. Hence G is 3-colorable.

If NOT-10-$COL \in P$ then 10-COL is in P. Since 3-$COL \le_m^p$ 10-COL we have that 3-COL is in P. Since 3-COL is NP-complete we have that P=NP.

8. (15 point) Show that if $B \in NP$ and $A \le_m^p B$ then $A \in NP$.

SOLUTION TO PROBLEM 8

Since $B \in NP$ there is a set C such that

$$B = \{x : (\exists^p y)[C(x,y)]\}.$$

Since $B \le_m^p A$ there is a function f such that

$$x \in A \text{ iff } f(x) \in B.$$

Hence note that

$$x \in A \text{ iff } f(x) \in B \text{ iff } (\exists^p y)[C(f(x),y)].$$

Hence we have

$$A = \{x : (\exists^p y)[C(f(x),y)]\}.$$

Let

$$C' = \{(x, y) \,:\, C(f(x), y)\}.$$

Note that $C' \in P$ and

$$A = \{x \,:\, (\exists^p y)[C'(x, y)]\}.$$

Hence $A \in NP$.