

## Combinatorial Arguments with Abbott and Costello

**Abbott:** Lou, how many subsets are there of  $\{1, \dots, n\}$ ?

**Costello:** Oh. You can either choose 0 elements, or choose 1 element, or choose 2 elements, etc. So the answer is  $\sum_{i=0}^n \binom{n}{i}$ .

**Abbott:** Well ... let me show you a different way to do it. The number 1 is either in the set  $A$  or not, so that's 2 choices. Then the number 2 is either in the set  $A$  or not, so that's 2 choices, etc. So the final answer is  $2 \times \dots \times 2 = 2^n$ . So, Costello, you did the problem your way, I did it my way, and we got different answers. What can you conclude?

**Costello:** That one of us is wrong?

**Abbott:** No. We've shown.  $\sum_{i=0}^n \binom{n}{i} = 2^n$ .

**Costello:** Really! I don't believe that! Prove it!!

**Abbott:** We did!

**Costello:** When?

**Abbott:** Just now.

**Costello:** What!?

**Abbott:** Whats on Second.

**Costello:** Who?

**Abbott:** Who's on first.

**Costello:** (Ignoring reference) Usually when I do a math problem two ways and get two answers I assume one of them is wrong and try to find my error. Its better than what a friend of mine did in elementary algebra— do a problem three times and then take the average.

**Abbott:** In math you can sometimes prove that two things are the same by solving the same problem two different ways.

**Costello:** No way!

**Abbott:** Way!

**Costello:** I'd like to know more about this.

**Abbott:** Okay. Here is one involving Fibonacci Numbers. Recall that the Fibonacci numbers are defined by

$$\begin{aligned}FIB(0) &= 1 \\FIB(1) &= 2 \\FIB(d) &= F(d-1) + F(d-2)\end{aligned}$$

**Costello:** I've seen those!

**Abbott:** Ah, but have you seen this recurrence:

$$\begin{aligned}F(0) &= 1 \\F(1) &= 2 \\F(d) &= 1 + F(0) + \sum_{i=2}^d F(i-2)\end{aligned}$$

**Costello:** Why should I care?

**Abbott:** Because it comes up in the analysis of Fib Heaps?

**Costello:** Why should I care about Fib Heaps?

**Abbott:** Assume, by induction, that  $n$  people care about Fib Heaps.

**Costello:** Alright, never mind, so you have two different recurrences, so what?

**Abbott:** Are they different?

**Costello:** Well, Yes, just look at them!

**Abbott:** Lets compute some values (They do so on the board and they are the same)

**Costello:** WOW, they are the same. Will you prove that? Or have you already somehow?

**Abbott:** Yes, I will prove it.

GOTO NOTES ON AMORTIZED ANALYSIS

**Costello:** That was fun! Do you have more?

**Abbott:** (sarcastic) Does a Chicken have lips?

**Costello:** (serious) Actually, does a chicken have lips?

**Abbott:** Er, never mind. Here is a nice lemma that will help us count how many elements in the  $n$ th row of Pascal's triangle are odd.

**Lemma:** For all  $k \leq n$  the following hold.

1.  $\binom{2n}{2k} \equiv \binom{n}{k} \pmod{2}$ .
2.  $\binom{2n+1}{2k+1} \equiv \binom{n}{k} \pmod{2}$ .
3.  $\binom{2n+1}{2k} \equiv \binom{n}{k} \pmod{2}$ .
4.  $\binom{2n}{2k+1} \equiv 0 \pmod{2}$ .

Hence, mod 2,

$$\binom{n}{k} \equiv \begin{cases} \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor} & \text{if } n \equiv 0 \text{ and } k \equiv 0 ; \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor} & \text{if } n \equiv 1 \text{ and } k \equiv 0 ; \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor} & \text{if } n \equiv 1 \text{ and } k \equiv 1 ; \\ 0 & \text{if } n \equiv 0 \text{ and } k \equiv 1 . \end{cases}$$

Note that the operator that maps  $n$  to  $\lfloor n/2 \rfloor$  just removes the last bit. Note that in three of the four cases the last bit is removed from  $n$  and  $k$ .

**Proof:** We prove the first item; the rest are similar.

How many strings are there in  $\{0,1\}^{2n}$  that have exactly  $2k$  ones? The answer is  $\binom{2n}{2k}$ . How many of these are palindromes? If you determine the  $k$  ones in the first  $n$  positions then the rest of the string is determined, so the answer is  $\binom{n}{k}$ . Let  $NONPAL(2n, 2k)$  be the number of nonpalindromes in  $\{0,1\}^{2n}$  that have  $2k$  ones. Hence

$$\binom{2n}{2k} = \binom{n}{k} + NONPAL(2n, 2k)$$

Recall that if  $x$  is a string then  $x^R$  is the string reversed. For every  $x \in NONPAL(2n, 2k)$  we have that  $x^R \in NONPAL(2n, 2k)$  and that  $x \neq x^R$ . Hence  $NONPAL(2n, 2k)$  is even. Therefore

$$\binom{2n}{2k} \equiv \binom{n}{k} \pmod{2}$$

**End of Proof**

By the Lemma, the following is true if all we care about is parity of  $\binom{n}{k}$ :

1. if  $n$  ends in 0 and  $k$  ends in 0, we can delete both of those bits,
2. if  $n$  ends in 1 and  $k$  ends in 0, we can delete both of those bits,
3. if  $n$  ends in 1 and  $k$  ends in 1, we can delete both of those bits,

Lets say I want to know all the numbers  $i$  such that  $\binom{76}{i}$  is odd. Lets look at  $\binom{76}{52}$ . In base 2 this is

$$\binom{1001100_2}{0110100_2} \equiv \binom{100110_2}{011010_2} \equiv \binom{10011_2}{01101_2} \equiv \binom{1001_2}{0110_2} \equiv \binom{100_2}{011_2} \pmod{2}.$$

We now have the one case where you cannot delete anything; however, by our Lemma we know

$$\binom{100_2}{011_2} \equiv 0 \pmod{2}.$$

The only way that  $\binom{76}{i}$  will be even is if there is a bit-place  $j$  such that 76 in base 2 has as its  $j$ th bit 0, and  $i$  has as its  $j$ th bit, 1. Hence we know that, for any choice of  $a, b, c, d \in \{0, 1\}$  the number

$$\binom{1001100_2}{a00bc00_2} \equiv 1 \pmod{2}.$$

We also know that any value of  $i$  that is not of the form  $a00bc00$  will cause  $\binom{1001100_2}{i} \equiv 1 \pmod{2}$ . There are  $2^3 = 8$  choices for  $a, b, c$ . Hence there are 8 values of  $i$  such that  $\binom{76}{i}$  is odd.

The above reasoning generalizes to the following theorem.

**Theorem:** The number of odd elements in the  $n$ th row of Pascals triangle (that is, the number of odd elements in  $\{\binom{n}{0}, \dots, \binom{n}{n}\}$ ) is  $2^m$  where  $m$  is the number of 1's in the binary expansion of  $n$ .

**Costello:** Wow! Do you have any that involve the Stirling Numbers?

**Abbott:** Do I ever!

Recall that the Stirling Number  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$  is the number of ways to place  $n$  people at  $k$  identical round tables (no table is empty). Also recall that  $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ .

**Costello:** How are you going to relate the Stirling Numbers to the Harmonic Numbers? The Stirling numbers are integers while the Harmonic Numbers are not.

**Abbott:** Have patience.

**Theorem:**  $\left[ \begin{smallmatrix} n+1 \\ 2 \end{smallmatrix} \right] = n!H_n$ .

**Proof:** How many ways can you put  $n + 1$  people at two identical round tables? One answer is  $\left[ \begin{smallmatrix} n+1 \\ 2 \end{smallmatrix} \right]$ . Another answer is as follows: Each person is a number. We will call the table with the number 1 at it the Left Table. We will represent the left table by  $(1, x_1, \dots, x_m)$ . So the question is, how many ways can we fill in the rest of the left table, and all of the right table? What if there were exactly  $i \geq 1$  at the right table? (Recall that no table is empty so we need not consider the  $i = 0$  case.) Then there would be  $n - i$  at the left table (we do not count person 1 who is already there), So we would choose  $i$  for the right table, which we can do in  $\binom{n}{i}$  ways, arrange them at the table, which we can do in  $(i - 1)!$  ways (the answer is not  $i!$  since the tables are round), then arrange the  $n - i$  at the left table anyway you like, which is  $(n - i)!$  ways. So the answer is  $\frac{n!}{i!(n-i)!}(i - 1)!(n - i)!$ .

$$\sum_{i=1}^n \frac{n!}{i!(n-i)!}(i - 1)!(n - i)! = \sum_{i=1}^n \frac{n!}{i!}(i - 1)! = \sum_{i=1}^n \frac{n!}{i} = n! \sum_{i=1}^n \frac{1}{i} = n!H_n.$$

**End of Proof**