

Combinatorial Arguments with Abbott and Costello

Abbott: Lou, how many subsets are there of $\{1, \dots, n\}$?

Costello: Oh. You can either choose 0 elements, or choose 1 element, or choose 2 elements, etc. So the answer is $\sum_{i=0}^n \binom{n}{i}$.

Abbott: Well ... let me show you a different way to do it. The number 1 is either in the set A or not, so that's 2 choices. Then the number 2 is either in the set A or not, so that's 2 choices, etc. So the final answer is $2 \times \dots \times 2 = 2^n$. So, Costello, you did the problem your way, I did it my way, and we got different answers. What can you conclude?

Costello: That one of us is wrong?

Abbott: No. We've shown. $\sum_{i=0}^n \binom{n}{i} = 2^n$.

Costello: Really! I don't believe that! Prove it!!

Abbott: We did!

Costello: When?

Abbott: Just now.

Costello: What!?

Abbott: Whats on Second.

Costello: Who?

Abbott: Who's on first.

Costello: (Ignoring reference) Usually when I do a math problem two ways and get two answers I assume one of them is wrong and try to find my error. Its better than what a friend of mine did in elementary algebra— do a problem three times and then take the average.

Abbott: In math you can sometimes prove that two things are the same by solving the same problem two different ways.

Costello: No way!

Abbott: Way!

Costello: I'd like to know more about this.

Abbott: Okay. Here is one involving Fibonacci Numbers. Recall that the Fibonacci numbers are defined by

$$\begin{aligned}FIB(0) &= 1 \\FIB(1) &= 2 \\FIB(d) &= F(d-1) + F(d-2)\end{aligned}$$

Costello: I've seen those!

Abbott: Ah, but have you seen this recurrence:

$$\begin{aligned}F(0) &= 1 \\F(1) &= 2 \\F(d) &= 1 + F(0) + \sum_{i=2}^d F(i-2)\end{aligned}$$

Costello: Why should I care?

Abbott: Because it comes up in the analysis of Fib Heaps?

Costello: Why should I care about Fib Heaps?

Abbott: Assume, by induction, that n people care about Fib Heaps.

Costello: Alright, never mind, so you have two different recurrences, so what?

Abbott: Are they different?

Costello: Well, Yes, just look at them!

Abbott: Lets compute some values (They do so on the board and they are the same)

Costello: WOW, they are the same. Will you prove that? Or have you already somehow?

Abbott: Yes, I will prove it.

GOTO NOTES ON AMORTIZED ANALYSIS

Costello: That was fun! Do you have more?

Abbott: (sarcastic) Does a Chicken have lips?

Costello: (serious) Actually, does a chicken have lips?

Abbott: Er, never mind. Here is a nice lemma that will help us count how many elements in the n th row of Pascal's triangle are odd.

Lemma: For all $k \leq n$ the following hold.

1. $\binom{2n}{2k} \equiv \binom{n}{k} \pmod{2}$.
2. $\binom{2n+1}{2k+1} \equiv \binom{n}{k} \pmod{2}$.
3. $\binom{2n+1}{2k} \equiv \binom{n}{k} \pmod{2}$.
4. $\binom{2n}{2k+1} \equiv 0 \pmod{2}$.

Hence, mod 2,

$$\binom{n}{k} \equiv \begin{cases} \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor} & \text{if } n \equiv 0 \text{ and } k \equiv 0 ; \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor} & \text{if } n \equiv 1 \text{ and } k \equiv 0 ; \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor} & \text{if } n \equiv 1 \text{ and } k \equiv 1 ; \\ 0 & \text{if } n \equiv 0 \text{ and } k \equiv 1 . \end{cases}$$

Note that the operator that maps n to $\lfloor n/2 \rfloor$ just removes the last bit. Note that in three of the four cases the last bit is removed from n and k .

Proof: We prove the first item; the rest are similar.

How many strings are there in $\{0,1\}^{2n}$ that have exactly $2k$ ones? The answer is $\binom{2n}{2k}$. How many of these are palindromes? If you determine the k ones in the first n positions then the rest of the string is determined, so the answer is $\binom{n}{k}$. Let $NONPAL(2n, 2k)$ be the number of nonpalindromes in $\{0,1\}^{2n}$ that have $2k$ ones. Hence

$$\binom{2n}{2k} = \binom{n}{k} + NONPAL(2n, 2k)$$

Recall that if x is a string then x^R is the string reversed. For every $x \in NONPAL(2n, 2k)$ we have that $x^R \in NONPAL(2n, 2k)$ and that $x \neq x^R$. Hence $NONPAL(2n, 2k)$ is even. Therefore

$$\binom{2n}{2k} \equiv \binom{n}{k} \pmod{2}$$

End of Proof

By the Lemma, the following is true if all we care about is parity of $\binom{n}{k}$:

1. if n ends in 0 and k ends in 0, we can delete both of those bits,
2. if n ends in 1 and k ends in 0, we can delete both of those bits,
3. if n ends in 1 and k ends in 1, we can delete both of those bits,

Lets say I want to know all the numbers i such that $\binom{76}{i}$ is odd. Lets look at $\binom{76}{52}$. In base 2 this is

$$\binom{1001100_2}{0110100_2} \equiv \binom{100110_2}{011010_2} \equiv \binom{10011_2}{01101_2} \equiv \binom{1001_2}{0110_2} \equiv \binom{100_2}{011_2} \pmod{2}.$$

We now have the one case where you cannot delete anything; however, by our Lemma we know

$$\binom{100_2}{011_2} \equiv 0 \pmod{2}.$$

The only way that $\binom{76}{i}$ will be even is if there is a bit-place j such that 76 in base 2 has as its j th bit 0, and i has as its j th bit, 1. Hence we know that, for any choice of $a, b, c, d \in \{0, 1\}$ the number

$$\binom{1001100_2}{a00bc00_2} \equiv 1 \pmod{2}.$$

We also know that any value of i that is not of the form $a00bc00$ will cause $\binom{1001100_2}{i} \equiv 1 \pmod{2}$. There are $2^3 = 8$ choices for a, b, c . Hence there are 8 values of i such that $\binom{76}{i}$ is odd.

The above reasoning generalizes to the following theorem.

Theorem: The number of odd elements in the n th row of Pascals triangle (that is, the number of odd elements in $\{\binom{n}{0}, \dots, \binom{n}{n}\}$) is 2^m where m is the number of 1's in the binary expansion of n .

Costello: Wow! Do you have any that involve the Stirling Numbers?

Abbott: Do I ever!

Recall that the Stirling Number $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ is the number of ways to place n people at k identical round tables (no table is empty). Also recall that $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.

Costello: How are you going to relate the Stirling Numbers to the Harmonic Numbers? The Stirling numbers are integers while the Harmonic Numbers are not.

Abbott: Have patience.

Theorem: $\left[\begin{smallmatrix} n+1 \\ 2 \end{smallmatrix} \right] = n!H_n$.

Proof: How many ways can you put $n + 1$ people at two identical round tables? One answer is $\left[\begin{smallmatrix} n+1 \\ 2 \end{smallmatrix} \right]$. Another answer is as follows: Each person is a number. We will call the table with the number 1 at it the Left Table. We will represent the left table by $(1, x_1, \dots, x_m)$. So the question is, how many ways can we fill in the rest of the left table, and all of the right table? What if there were exactly $i \geq 1$ at the right table? (Recall that no table is empty so we need not consider the $i = 0$ case.) Then there would be $n - i$ at the left table (we do not count person 1 who is already there), So we would choose i for the right table, which we can do in $\binom{n}{i}$ ways, arrange them at the table, which we can do in $(i - 1)!$ ways (the answer is not $i!$ since the tables are round), then arrange the $n - i$ at the left table anyway you like, which is $(n - i)!$ ways. So the answer is $\frac{n!}{i!(n-i)!}(i - 1)!(n - i)!$.

$$\sum_{i=1}^n \frac{n!}{i!(n-i)!}(i - 1)!(n - i)! = \sum_{i=1}^n \frac{n!}{i!}(i - 1)! = \sum_{i=1}^n \frac{n!}{i} = n! \sum_{i=1}^n \frac{1}{i} = n!H_n.$$

End of Proof