CMSC 651 Advanced Algorithms	Lecture 22
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1 The Primal-Dual Method for Linear Programming

We will outline a framework for solving linear programs. While this framework does not offer an efficient algorithm for linear programming, when we apply this method to specific optimization problems, it yields efficient algorithms. For example, we will show how to apply this method to *derive* the Hungarian method for solving the maximum weight perfect matching problem in bipartite graphs. This is really nice, because most algorithmic development tends to be adhoc and non-systematic, requiring brilliant flashes of insight.

First we consider a linear program in the following form (with n variables and m constraints).

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \text{ for } 1 \leq i \leq m$$

$$x_j \geq 0 \text{ for } 1 \leq j \leq n$$

The dual of this LP can be written in a simple manner as follows (with m variables and n constraints):

$$\min \pi' \cdot b$$

$$\pi' A \geq c$$

$$\pi' \text{ unrestricted}$$

Suppose we have a feasible dual solution π. Recall that the complementary slackness conditions tell us that at optimality, either a primal variable is exactly 0, or the corresponding dual constraint is met with equality. At the same time, either a dual variable is exactly 0, or the corresponding primal constraint is met with equality, Moreover, these are both necessary and sufficient conditions for optimality.

In our situation, if we impose these conditions we can observe the following. Let $J_\pi = \{j : \pi A_j = c_j\}$, where A_j is the j^{th} column of the matrix A. In other words, this is exactly the set of dual constraints that are tight. In other words, a feasible solution x is optimal if and only if it is feasible, and $x_j = 0$ when $j \notin J_\pi$.

The search for such a feasible x can be used to simplify the original LP significantly. We can write the original LP as a simpler LP problem P'.

$$\max \sum_{j \in J_\pi} c_j x_j$$

$$\sum_{j \in J_\pi} a_{ij} x_j = b_i \text{ for } 1 \leq i \leq m$$

$$x_j \geq 0 \text{ for } j \in J_\pi$$

If we can find a feasible solution for P', then we have an optimal solution, since it definitely satisfies the complementary slackness conditions since $x_j = 0$ when $j \notin J_\pi$.

The search for such a feasible solution can be written as another LP (called the restricted primal (RP)) in the following form. We introduce m non-negative variables. Notice that obtaining a feasible solution here is quite easy if the b vector is non-negative, since we can set all original variables to 0 and assign $x_i^a = b_i$. In fact we can obtain an optimal solution by using the simplex method. The main point is that the cost vector c has now vanished completely! This system of equations is often so simple that it can be solved manually as well (sometimes without using the simplex method). The RP is defined as follows.

$$\min \sum_{i=1}^{m} x_i^a$$

$$\sum_{j \in J_\pi} a_{ij} x_j + x_i^a = b_i \text{ for } 1 \le i \le m$$

$$x_j \ge 0 \text{ for } j \in J_\pi$$

$$x_i^a \ge 0 \text{ for } 1 \le i \le m$$

Note that it is easy to prove the following. P' **has a feasible solution if and only if the optimal value of** RP **is exactly 0.** We can solve RP and determine the value of the optimal solution. If it is exactly 0, then the optimal solution for RP, is in fact an optimal solution for P, since it is feasible for P'. If the optimal solution for RP is non-zero, then it must be strictly positive. In this case we will modify the dual feasible solution π to compute a new set of indices J_π. In the process, we will show that the value of the dual solution strictly decreases.

Let the Dual of the RP, be DRP.

$$\max \pi b$$

$$\pi_i \le 1 \text{ for } 1 \le i \le m$$

$$\pi A_j \le 0 \text{ for } j \in J_\pi$$

Let $\overline{\pi}$ be an optimal solution for DRP. Note that $\overline{\pi} b > 0$ (this is because the optimal value of RP is non-zero and this is the same as the optimal value of DRP). We change π as follows. Let $\pi^* = \pi - \theta\overline{\pi}$ be the new dual solution for some $\theta > 0$. Note that $\pi^* b = \pi b - \theta \overline{\pi} b$. Since the second term is positive, the dual objective function value decreases. We now show how to maintain dual feasibility.

At the same time if $j \in J_\pi$ then $\pi^* A_j \ge \pi A_j$, hence the dual constraints are all satisfied. If for all $j \notin J_\pi$ we have $\overline{\pi} A_j \le 0$, again the dual constraints are satisfied. In this case we can increase θ to make the dual solution decrease as much as we want. This implies that the original LP instance was not feasible. For $j \notin J_\pi$ in case $\overline{\pi} A_j > 0$ then this limits the increase in θ. In fact, the maximum value we can give θ without violating the j^{th} dual constraints is

$$\theta_j = \frac{\pi A_j - c_j}{\overline{\pi} A_j} \text{ for } j \notin J_\pi, \text{ such that } \overline{\pi} A_j > 0.$$

We pick $\theta = \min_{j \notin J_\pi} \theta_j$. In this way at least one new index will enter the set J_π. It may be that some indices leave J_π as well. We update the dual solution from π to π^* and then repeat the entire process, until the DRP has a zero value solution and we can stop. At this stage we have primal and dual feasible solutions that satisfy complementary slackness.