Promise Problem AM Example
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1 A Promise Problem Example

We present a promise problem example of something in AM.
PROMISE: ϕ is a formula on n variables. Either $\#(\phi) \geq 2^{n-\sqrt{n}}$ or $\#(\phi) \leq 2^{\sqrt{n}}$.
PROBLEM: Merlin wants to convince Arthur that $\#(\phi) \geq 2^{n-\sqrt{n}}$

We give a protocol in terms of Arthur and Merlin. It can be rephrased formally.

Here is what we want:

- If $\#(\phi) \geq 2^{n-\sqrt{n}}$ then Prob that Merlin can respond with a y that convinces Arthur that $\#(\phi) \geq 2^{n-\sqrt{n}}$ is $\geq \frac{3}{4}$
- If $\#(\phi) \leq 2^{\sqrt{n}}$ then Prob that Merlin can respond with a y that convinces Arthur that $\#(\phi) \geq 2^{n-\sqrt{n}}$ is $\leq \frac{1}{4}$

1. Input(ϕ). Both Arthur and Merlin see ϕ. Let n be the number of variables in ϕ.
2. Arthur sends Merlin kn random bits that form a $k \times n$ matrix H. (We will pick k later. It will be a function of n.)
3. Merlin responds with a y (if there is a y such that $\phi(y) = T$ and $H(y) = 0$ then this is what Merlin will send.)
4. Arthur evaluates $\phi(y)$ and $H(y)$. If $\phi(y) = T$ and $H(y) = 0$ then ACCEPT, otherwise REJECT.

We need to pick a value of k such that

We first find conditions on k such that the following is true:

if $\#(\phi) \geq 2^{n-\sqrt{n}}$ *then Prob that Merlin can respond with a* y *that satisfies Arthur is* $\geq \frac{3}{4}$.

Let

$$X = |\{y : \phi(y) = T \wedge H(y) = 0\}|.$$

If $\#(\phi) \geq 2^{n-\sqrt{n}}$ then

- $E(X) = 2^{-k}\#\phi \geq 2^{n-\sqrt{n}-k}$.
- $Var(X) \leq 2^{-k}\#\phi \leq 2^{n-k}$

We need that the probability that $X = 0$ is small. If $X = 0$ then

$$E(X) - X = E(X) \geq 2^{n-\sqrt{n}-k}.$$

Hence (trivially)

$$|E(X) - X| \geq 2^{n-\sqrt{n}-k}.$$

The prob of this happening we can bound using Chebyshev's inequality:

$$\begin{aligned} \Pr(|E(X) - X| \geq 2^{n-\sqrt{n}-k}) &\leq \frac{Var(X)}{(2^{n-\sqrt{n}-k})^2} \\ &\leq \frac{2^{n-k}}{(2^{n-\sqrt{n}-k})^2} \\ &\leq \frac{2^{n-k}}{2^{2n-2\sqrt{n}-2k}} \\ &\leq 2^{-n+k+2\sqrt{n}} \end{aligned}$$

We need this to be $\leq 1/4$. Hence we need

$$2^{-n+k+2\sqrt{n}} \leq 2^{-2}$$

$$-n + k + 2\sqrt{n} \leq -2$$

$$k \leq n - 2\sqrt{n} - 2$$

We now find conditions of k such that the following is true:
If $\#(\phi) \leq 2^{\sqrt{n}}$ then Prob that Merlin can respond with a y that satisfies Arthur is $\leq \frac{1}{4}$.

Again let

$$X = |\{y : \phi(y) = T \wedge H(y) = 0\}|.$$

If $\#(\phi) \leq 2^{\sqrt{n}}$ then

- $E(X) = 2^{-k}\#\phi \leq 2^{\sqrt{n}-k}$.
- $Var(X) \leq 2^{-k}\#\phi \leq 2^{\sqrt{n}-k}$.

We need that the probability that $X \geq 1$ is small. If $X \geq 1$ then

$$X - E(X) \geq 1 - 2^{\sqrt{n}-k}.$$

Hence (trivially)

$$|E(X) - X| \geq 1 - 2^{\sqrt{n}-k}.$$

This is clumsy to work with, but note that it implies

$$|E(X) - X| \geq 1/2.$$

We bound the prob of this happening using Chebyshev's inequality. (We can also do this one with Markov's inequality which we will present later.)

$$\begin{aligned} \Pr(|E(X) - X| \geq 1/2) &\leq \tfrac{Var(X)}{(1/2)^2} \\ &\leq \tfrac{2^{\sqrt{n}-k}}{1/4} \\ &\leq 2^{\sqrt{n}-k+2} \end{aligned}$$

We need this to be $\leq 1/4$. Hence we need

$$2^{\sqrt{n}-k+2} \leq 2^{-2}$$

$$\sqrt{n} - k + 2 \leq -2$$

$$\sqrt{n} - k \leq -4$$

$$k \geq \sqrt{n} + 4$$

SO we have two contraints on k, we need

$$k \leq n - 2\sqrt{n} - 2$$

and

$$k \geq \sqrt{n} + 4$$

We can satisfy these both easily with, say, $k = n - 3\sqrt{n}$.

The last one we can also do with Markov's inequality. Recall that Markov's inequality is

$$\Pr(X \geq k) \leq E(X)/k$$

In our case we need

$$\Pr(X \geq 1) \leq E(X)/1 = 2^{\sqrt{n}-k}$$

So we need

$$2^{\sqrt{n}-k} \leq 2^{-2}$$

$$\sqrt{n} - k \leq -2$$

$$k \geq \sqrt{n} + 2$$

.

And again $k = n - 3\sqrt{n}$ works.