

1 Broad overview:

Let $S_n = \{\pi_1, \dots, \pi_{n!}\}$.

Let $\pi_i \text{Aut}(G) = \{\pi_i \sigma \mid \sigma \in \text{Aut}(G)\}$ for each i .

We're going to show that the sets $\pi_i \text{Aut}(G)$ are each of size $|\text{Aut}(G)|$ and that any two of them are disjoint or equal. This will show that S_n is partitioned into $\frac{n!}{|\text{Aut}(G)|}$ sets of the form $\pi_i \text{Aut}(G)$. We will then show that all members of $\pi_i \text{Aut}(G)$ have the same effect on G , and that elements of different equivalence classes never have the same effect on G . This will show G is isomorphic to exactly $\frac{n!}{|\text{Aut}(G)|}$ different graphs.

2 Proof that the sets $\pi_i \text{Aut}(G)$ are of size $|\text{Aut}(G)|$ and partition S_n

2.1 $|\pi_i \text{Aut}(G)| = |\text{Aut}(G)|$

Suppose $\pi_i \sigma = \pi_i \tau$ for some $\sigma, \tau \in \text{Aut}(G)$.

Then $\pi_i^{-1} \pi_i \sigma = \pi_i^{-1} \pi_i \tau$ (note that permutations have inverses).

So $\sigma = \tau$.

Therefore, the map $f : \text{Aut}(G) \rightarrow \pi_i \text{Aut}(G)$ defined by $f(\sigma) = \pi_i \sigma$ is one to one. Since it's obviously onto, it's a bijection.

So $|\pi_i \text{Aut}(G)| = |\text{Aut}(G)|$.

2.2 The $\pi_i \text{Aut}(G)$'s partition S_n

Suppose $\sigma \in \pi_i \text{Aut}(G)$ and $\sigma \in \pi_j \text{Aut}(G)$. We will show this means $\pi_i \text{Aut}(G) = \pi_j \text{Aut}(G)$.

We have $\sigma = \pi_i \tau_i = \pi_j \tau_j$ for some $\tau_i, \tau_j \in \text{Aut}(G)$.

$$\pi_i \tau_i = \pi_j \tau_j$$

$$\text{So } \pi_i \tau_i \tau_j^{-1} = \pi_j.$$

So for all $\psi \in \text{Aut}(G)$, $\pi_j \psi = \pi_i \tau_i \tau_j^{-1} \psi \in \pi_i \text{Aut}(G)$ (inverses and compositions of automorphisms are automorphisms)

So $\pi_j \text{Aut}(G) \subseteq \pi_i \text{Aut}(G)$.

A symmetric argument shows that $\pi_j \text{Aut}(G) = \pi_i \text{Aut}(G)$.

Note that any element of S_n is in some $\pi_i \text{Aut}(G)$ because $\pi_i \in \pi_i \text{Aut}(G)$ since the identity map is an automorphism.

So the $\pi_i \text{Aut}(G)$'s partition S_n .

Let k be the number of distinct $\pi_i \text{Aut}(G)$ s. Then $k |\text{Aut}(G)| = |S_n|$

So there must be $k = \frac{|S_n|}{|\text{Aut}(G)|} = \frac{n!}{|\text{Aut}(G)|}$ distinct $\pi_i \text{Aut}(G)$'s.

3 Proof that all elements of $\pi_i \text{Aut}(G)$ have the same effect on G , and elements of different $\pi_i \text{Aut}(G)$'s have different effect on G .

3.1 $\phi, \psi \in \pi_i \text{Aut}(G)$ implies $\phi G = \psi G$

If $\phi, \psi \in \pi_i \text{Aut}(G)$, then $\phi = \pi_i \sigma$ and $\psi = \pi_i \tau$ for some $\sigma, \tau \in \text{Aut}(G)$.

So $\phi G = \pi_i \sigma G = \pi_i G$ since $\sigma \in \text{Aut}(G)$.

Similarly, $\psi G = \pi_i \tau G = \pi_i G$ since $\tau \in \text{Aut}(G)$.

So $\phi G = \psi G$

3.2 If $\phi \in \pi_i \mathbf{Aut}(G)$ and $\psi \in \pi_j \mathbf{Aut}(G)$ where $\pi_i \mathbf{Aut}(G) \neq \pi_j \mathbf{Aut}(G)$ then $\phi G \neq \psi G$

Assume $\phi \in \pi_i \mathbf{Aut}(G)$ and $\psi \in \pi_j \mathbf{Aut}(G)$ and $\phi G = \psi G$.

Then $\phi = \pi_i \sigma$ and $\psi = \pi_j \tau$ for $\sigma, \tau \in G$

So $\pi_i \sigma G = \pi_j \tau G$.

So $\pi_i G = \pi_j G$.

So $\pi_j^{-1} \pi_i G = G$ (we can do this safely because $(\pi_k \pi_\ell) G = \pi_k (\pi_\ell G)$).

So $\pi_j^{-1} \pi_i \in \mathbf{Aut}(G)$.

So $\pi_j (\pi_j^{-1} \pi_i) = \pi_i \in \pi_j \mathbf{Aut}(G)$.

Since the sets are disjoint or equal, $\pi_j \mathbf{Aut}(G) = \pi_i \mathbf{Aut}(G)$.

4 Wrap up

Clearly any graph with vertices $\{1, \dots, n\}$ that is isomorphic to G is $\pi_i G$ for some $\pi_i \in S_n$.

Since each of the $\frac{n!}{|\mathbf{Aut}(G)|}$ equivalence classes acts on G differently, and each element of each equivalence class acts on G the same way, there are $\frac{n!}{|\mathbf{Aut}(G)|}$ graphs with vertices $\{1, \dots, n\}$ that are isomorphic to G .