Notes On Polynomial Hierarchy
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1 Defintions and Notation

Recall the definition of NP.

Def 1.1 $A \in$ NP if there exists a polynomial p and a polynomial prediate B such that
$A = \{x \mid (\exists y)[|y| \le p(|x|) \wedge B(x,y)]\}$.

We want to build on this, but first we need a notation

Notation 1.2

1. The expression
   $A = \{x \mid (\exists^p y)[B(x,y)]\}$
   means that there is a polynomial p such that
   $A = \{x \mid (\exists y, |y| \le p(|x|))[B(x,y)]\}$.
2. The expression
   $A = \{x \mid (\forall^p y)[B(x,y)]$
   means that there is a polynomial p such that
   $A = \{x \mid (\forall y, |y| \le p(|x|))[B(x,y)]\}$.
3. The expression
   $A = \{x \mid (\forall^p y)(\exists^p z)[B(x,y,z)]$
   means that there are polynomials p_1, p_2 such that
   $A = \{x \mid (\forall y, |y| \le p_1(|x|))(\exists z, |z| \le p_2(|x|))[B(x,y,z)]\}$.
4. One can define this notation for as long a string of quantifiers as you like. We leave the formal definition to the reader.

In the following definition we include a definition and an alternative definition.

Def 1.3

1. $A \in \Sigma_0^p$ if $A \in$ P. $A \in \Pi_0^p$ if $A \in$ P. (We include this so we use it inductively later.)

2. $A \in \Sigma_1^p$ if there exists a set $B \in$ P such that

   $A = \{x \mid (\exists^p y)[B(x,y)]\}$.

   This is just NP.

3. $A \in \Pi_1^p$ if there exists a set $B \in$ P such that

   $A = \{x \mid (\forall^p y)[B(x,y)]\}$.

   This is just all sets A such that $\overline{A} \in$ NP. It is often called co-NP.

4. $A \in \Sigma_2^p$ if there exists a set $B \in$ P such that

   $A = \{x \mid (\exists^p y)(\forall^p z)[B(x,y,z)]\}$.

5. $A \in \Sigma_2^p$ (alternative definition) if there exists a set $B \in \Pi_1^p$ such that

   $A = \{x \mid (\exists^p y)[B(x,y)]\}$.

6. $A \in \Pi_2^p$ if there exists a set $B \in$ P such that

   $A = \{x \mid (\forall^p y)(\exists^p z)[B(x,y,z)]\}$.

7. $A \in \Pi_2^p$ (alternative definition) if $\overline{A} \in \Sigma_2^p$.

8. Let $i \in \mathsf{N}$. If i is even then $A \in \Sigma_i^p$ if there exists $B \in$ P such that

   $A = \{x \mid (\exists^p y_1)(\forall^p y_2)\cdots(\forall^p y_i)[B(x,y_1,\ldots,y_i)]$

   If i is odd then $A \in \Sigma_i^p$ if there exists $B \in$ P such that

   $A = \{x \mid (\exists^p y_1)(\forall^p y_2)\cdots(\exists^p y_i)[B(x,y_1,\ldots,y_i)]$

9. Let $i \in \mathsf{N}$. If i is even then $A \in \Pi_i^p$ if there exists $B \in$ P such that

   $A = \{x \mid (\forall^p y_1)(\exists^p y_2)\cdots(\exists^p y_i)[B(x,y_1,\ldots,y_i)]$

   If i is odd then $A \in \Pi_i^p$ if there exists $B \in$ P such that

   $A = \{x \mid (\forall^p y_1)(\exists^p y_2)\cdots(\forall^p y_i)[B(x,y_1,\ldots,y_i)]$

10. Let $i \in \mathsf{N}$ and $i \geq 1$. $A \in \Sigma_i^p$ (alternative definition) if there exists $B \in \Pi_{i-1}^p$ such that

    $A = \{x \mid (\exists^p y)[B(x,y)]\}$.

    (Note- we use the definition of Σ_0^p, Π_0^p here.)

11. $A \in \Pi_i^p$ (alternative definition) if $\overline{A} \in \Sigma_i^p$.

12. The *polynomial hierarchy*, denoted PH, is $\bigcup_{i=0}^{\infty} \Sigma_i^{\mathrm{p}}$. Note that this is the same as $\bigcup_{i=0}^{\infty} \Pi_i^{\mathrm{p}}$.

Def 1.4 A set A is Σ_i^p-*complete* if both of the following hold.

1. $A \in \Sigma_i^p$, and
2. For all $B \in \Sigma_i^p$, $B \leq_{\mathrm{m}}^{\mathrm{p}} A$.

Def 1.5 A set A is Π_i^p-*complete* if both of the following hold.

1. $A \in \Pi_i^p$, and
2. For all $B \in \Pi_i^p$, $B \leq_{\mathrm{m}}^{\mathrm{p}} A$.

Def 1.6 A set A is Π_i^p-*complete* (Alternative Definition) if $\overline{A}$ is Σ_i^p-complete.

Example 1.7 In all of the examples below x and y and x_i are vectors of Boolean variables.

1. $A = \{\phi(x,y) \mid (\exists b)(\forall c)[\phi(b,c)]\}$. This set is Σ_2^p-complete. It is clearly in Σ_2^p. This is called QBF_2. The QBF stands for Quantified Boolean Formula. The proof that it is Σ_2^{p}-complete uses Cook's Theorem.

2. One can define QBF_i easily. It is Σ_i^p-complete.

3. QBF is the set of all $\phi(x_1,\ldots,x_n)$ (the x_i's are vectors of variables) such that $(\exists x_1)(\forall x_2)\cdots(Qx_n)[\phi(x_1,\ldots,x_n)]$. ($Q$ is $\exists^p$ if n is odd and is $\forall^p$ if n is even.) This set is thought to not be in any Σ_i^p or Π_i^p.

4. Let $TWO = \{\phi \mid \phi$ has exactly two satisfying assignments $\}$. We show that $TWO \in \Sigma_2^p$.

   $TWO =$

   $\{\phi \mid (\exists b, c)(\forall d)[b \neq c \wedge \phi(b) \wedge \phi(c) \wedge (\phi(d) \Rightarrow ((d = b) \vee (d = c)))\}$

   It is not known if TWO is Σ_2^p-complete; however it is thought to NOT be.

5. One can define $THREE$, $FOUR$, etc. easily. They are all in Σ_2^p.

6. One can define variants of TWO having to do with finding TWO hamiltonian cycles, TWO k-cliques, etc. Also $THREE$, etc. These are all Σ_2^p.

7. $ODD = \{\phi \mid \phi$ has an odd number of satisfying assignments $\}$ is thought to NOT be in PH.