A Sane Proof that $COL_k \le COL_3$

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1 Introduction

Let $A \le B$ mean that there is a polynomial-time computable function f such that $x \in A$ iff $f(x) \in B$.

Def 1.1 Let $k \ge 2$. COL_k is the set of all graphs that are k-colorable

The following are well known.

- For all $k \ge 2$, $COL_k \le SAT$ (this is by the Cook-Levin Theorem).
- For all $k \ge 2$, For all $k \ge 3$, $SAT \le COL_k$, hence COL_k is NP-complete.
- If $a < b$ then $COL_a \le COL_b$ by an easy reduction (Take G and add K_{b-a} and an edge from every elements of K_{b-a} to the original graph.)

The proof that $COL_3 \le COL_4$ is very easy: just add a vertex to G and connect it to all the elements of G. Is $COL_4 \le COL_3$? Yes via

$$COL_4 \le SAT \le COL_3.$$

This is true but unsatisfying. One of my students said

It's counterintuitive and makes me sad.

So we asked informally: Is there a SANE reduction $COL_4 \le COL_3$. There is and we present it here. In fact we show $COL_k \le COL_3$.

A sane proof is already known. Let $HCOL_k$ is the set of all hypergraphs that are k-colorable Lovasz [**?**] showed

$$COL_k \leq HCOL_2 \leq COL_3.$$

Our proof does not use $HCOL_2$ or any $HCOL_k$.

Def 1.2 $GAD(x, y, z)$ is the following graph. (The vertices that don't have labels are never referred to so we don't need to label them.)

```
x          y
|          |
|          |
|          |
o - - - o
\          /
 \        /
  \    /
   y
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We leave the proof of the following easy lemma to the reader.

Lemma 1.3 *If $GAD(x, y, z)$ is three colored and x, y get the same color, then z also gets that color.*

Def 1.4 $GAD(x_1, \ldots, x_k, z)$ consists of $GAD(x_1, x_2, y_1)$, $GAD(y_1, x_3, y_2)$, $GAD(y_2, x_4, y_3)$, …, $GAD(y_{k-3}, x_{k-1}, y_{k-2})$, and $GAD(y_{k-2}, x_k, z)$. Note that, (1) not including $x_1, \ldots, x_k, z$, $GAD(y_{k-2}, x_k, z)$ has $3(k-2) + 1 = 3k - 5 \leq 3k$ vertices, and (2) $5(k-1) = 5k - 5 \leq 5k$ edges.

We leave the proof of the following easy lemma to the reader.

Lemma 1.5 *Let $k \geq 2$. If $GAD(x_1, x_2, \ldots, x_k, z)$ is three colored and $x_1, \ldots, x_k$ get the same color, then z also gets that color.*

Theorem 1.6 *Let $k \geq 2$. $COL_k \leq COL_3$ by a simple reduction. Let f be the reduction. If G has n vertices and e edges then $f(G) = G'$ has $\leq 2k^2n + 2ke$ vertices and $\leq 3k^2n + 2ke$ edges.*

Proof: Let G have vertices $v_1, \ldots, v_n$ and edge set E. We construct G':

1. Vertices T, F, R which will form a triangle. In any coloring they have different colors which we call T, F, R. This is 3 vertices and 3 edges. (We won't count these in the end since our crude upper bounds on the vertices and edges in G' will clearly be over by at least 3.)

2. For $1 \leq i \leq n$ and $1 \leq j \leq k$ vertex v_{ij}. All of these will be connected by an edge to vertex R. This will be kn vertices and kn edges. Here is our intent and how we achieve it:

   (a) For all $1 \leq i \leq n$ our intent is: v_{ij} is colored T means that vertex v_i in G is colored j; v_{ij} is colored F means that vertex v_i in G is NOT colored j.

   (b) For all $1 \leq i \leq n$ we need that *at least one* of $v_{i1}, \ldots, v_{in}$ is colored T. Hence we need it to NOT be the case that $v_{i1}, v_{i2}, \ldots, v_{in}$ are all colored F. We place the gadget $G(v_{i1}, \ldots, v_{in}, T)$ in the graph. If $v_{i1}, \ldots, v_{in}$ are all colored F then this gadget will not be 3-colorable. This is $\leq 3kn$ vertices and $\leq 5kn$ edges.

   (c) For all $1 \leq i \leq n$ we need that *at most one* of $v_{i1}, \ldots, v_{ik}$ is colored T. Hence we need that for each pair at most one is colored T. For each $1 \leq j_1 < j_2 \leq k$ we place the gadget $GAD(v_{ij_1}, v_{ij_2}, F)$. This is $n\binom{k}{2} \times 2 \leq k^2n$ vertices and $n\binom{k}{2} \times 5 \leq 2.5k^2n$ edges.

3. For each edge (v_i, v_j) in the original graph we want to make sure that v_i and v_j are not the same color. Place the gadgets $GAD(v_{i1}, v_{j1}, F)$, $GAD(v_{i2}, v_{j2}, F)$,..., $GAD(v_{ik}, v_{jk}, F)$. This is $2ke$ vertices and $5ke$ edges.

Note that the number of vertices in G' is $\leq kn + 3kn + k^2n + 2ke \leq 2k^2n + 2ke$ vertices and $\leq kn + 5kn + 2.5k^2n + 2ke \leq 3k^2n + 2ke$ edges.

Clearly G is k-colorable iff G' is 3-colorable.

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