SOME EXTREME FORMS DEFINED IN TERMS OF ABELIAN GROUPS

E. S. BARNES and G. E. WALL

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1. Introduction

Let $f(\boldsymbol{x}) = f(x_1, x_2, \ldots, x_n)$ be a positive definite quadratic form of determinant D, and let M be the minimum of $f(\boldsymbol{x})$ for integral $\boldsymbol{x} \neq \boldsymbol{0}$. Then we set

$$(1.1) \qquad \gamma_n(f) = M/D^{1/n}$$

and

$$(1.2) \qquad \gamma_n = \max \gamma_n(f),$$

the maximum being over all positive forms f in n variables. f is said to be *extreme* if $\gamma_n(f)$ is a local maximum for varying f, *absolutely extreme* if $\gamma_n(f)$ is an absolute maximum, i.e. if $\gamma_n(f) = \gamma_n$.

It is well known that γ_n is of order n for large n; in fact, by the classical results of Blichfeldt and Hlawka (see [3], Ch. II, § 6),

$$(1.3) \qquad \frac{1}{2\pi e} \leqq \liminf \frac{\gamma_n}{n} \leqq \limsup \frac{\gamma_n}{n} \leqq \frac{1}{\pi e}.$$

On the other hand, no one has yet constructed a sequence of forms for which $\gamma_n(f)$ is unbounded, let alone of order n, as $n \to \infty$. We have therefore thought it worthwhile to describe, in some detail, a new class of extreme forms yielding values $\gamma_n(f)$ of order $n^{\frac{1}{2}}$ for suitable large n.

More specifically, corresponding to each $N = 2^n$ $(n = 2, 3, \ldots)$ and to each sequence of integers $\lambda_0, \lambda_1, \ldots, \lambda_n$ satisfying

$$\lambda_0 = 0,\ \lambda_r - 1 \leqq \lambda_{r-1} \leqq \lambda_r \quad (1 \leqq r \leqq n),$$

we construct a positive N-variable form $f_{(\lambda)}$, which we show to be extreme in most cases. We prove also that, for each N, there is an $f_{(\lambda)}$ satisfying $\gamma_N(f) = (\frac{1}{2}N)^{\frac{1}{2}}$, whence

$$(1.4) \qquad \gamma_N \geqq (\tfrac{1}{2}N)^{\frac{1}{2}} \text{ for } N = 2^n,\ n \geqq 2.$$

Our method of construction is based on the structure of the elementary Abelian group of order 2^n.

Further investigation shows that by an elaboration of the method, or by using a similar method based on Abelian groups of exponent 3, (1.4) can be strengthened for large N. It should also be noted that (1.4) is precise for $N = 4$ or 8, since $\gamma_4 = \sqrt{2}$, $\gamma_8 = 2$, and that, for sufficiently small N, (1.4) is a considerable improvement on known results.

The general properties of forms and lattices which we require are collected in § 2. The forms $f_{(\lambda)}$ and their lattices $\Lambda_{(\lambda)}$ are defined in § 3, the determinant D and minimum M are calculated and a criterion for the minimal vectors is given. In § 4, we prove that the $f_{(\lambda)}$ are extreme (under suitable conditions) and investigate the equivalences between them. In § 5, we enumerate the minimal vectors. A table of the inequivalent extreme forms $f_{(\lambda)}$ in 4, 8, 16 and 32 variables is given at the end of the paper.

2. Forms and lattices*

A positive quadratic form $f(\boldsymbol{x}) = f(x_1, x_2, \ldots, x_n)$ is said to have lattice Λ if

$$f(\boldsymbol{x}) = \xi'\xi,$$

where ξ runs through the points of Λ when $\boldsymbol{x}$ runs through all integral vectors; i.e. if

$$f(\boldsymbol{x}) = \boldsymbol{x}'A\boldsymbol{x} = \boldsymbol{x}'T'T\boldsymbol{x}$$

where Λ is specified by

$$\xi = T\boldsymbol{x}, \ \boldsymbol{x} \text{ integral.} \tag{2.1}$$

Then $D(f) = \det A = (\det T)^2 = d^2(\Lambda)$.

Clearly, if f has lattice Λ, it also has lattice $R\Lambda$, where R is any orthogonal transformation. Also, equivalent forms correspond to the same lattice; for if U is an integral unimodular transformation, then T and TU define the same lattice and correspond to the equivalent forms $\boldsymbol{x}'A\boldsymbol{x}$ and $\boldsymbol{x}'U'AU\boldsymbol{x}$. It is thus easy to see that two forms f_1, f_2 with lattices Λ_1, Λ_2 are equivalent if and only if $\Lambda_2 = R\Lambda_1$ for some orthogonal transformation R.

We define the reciprocal lattice Λ^{-1} of Λ to be the set of points $\boldsymbol{\eta} = T'^{-1}\boldsymbol{x}$, $\boldsymbol{x}$ integral, where Λ is given by (2.1). Then $d(\Lambda^{-1})d(\Lambda) = 1$, and the corresponding quadratic forms $\boldsymbol{x}'T'T\boldsymbol{x}$, and $\boldsymbol{x}'(T'T)^{-1}\boldsymbol{x}$ are reciprocal (i.e. have inverse matrices).

We shall be concerned in this paper only with forms whose lattices are sublattices of the integer lattice Γ. For these, it is convenient to define the notion of dual lattices modulo k.

Let k be a positive integer, and Λ_1, Λ_2 lattices such that

$$k\Gamma \subset \Lambda_1, \ \Lambda_2 \subset \Gamma.$$

* For further general information we refer the reader to Coxeter's paper [2].

Then Λ_2 is said to be the *dual* of Λ_1 modulo k if it consists of those $\boldsymbol{x} \in \Gamma$ satisfying

(2.2) $$\boldsymbol{x}'\boldsymbol{y} \equiv 0 \pmod{k} \text{ for all } \boldsymbol{y} \in \Lambda_1.$$

It is easy to see that in fact

(2.3) $$\Lambda_2 = k\Lambda_1^{-1}.$$

For if Λ_1 is defined by (2.1), i.e. $\Lambda_1 = T\Gamma$, then an integral $\boldsymbol{x} \in \Lambda_2$ if and only if

$$\boldsymbol{x}'T\boldsymbol{y} \equiv 0 \pmod{k} \text{ for all } \boldsymbol{y} \in \Gamma,$$

i.e. if and only if

(2.4) $$T'\boldsymbol{x} \in k\Gamma.$$

Now since $k\Gamma \subset \Lambda_1$, the relation $k\boldsymbol{u} = T\boldsymbol{x}$ has a solution $\boldsymbol{x} \in \Gamma$ for every $\boldsymbol{u} \in \Gamma$, so that kT^{-1} is an integral matrix. Hence (2.4) is equivalent to

$$\boldsymbol{x} \in kT'^{-1}\Gamma = k\Lambda_1^{-1}$$

and (2.3) is established.

From (2.3), we see that the relation between Λ_1 and Λ_2 is symmetrical, so that also Λ_1 is the dual of Λ_2 modulo k; and

$$d(\Lambda_1)\,d(\Lambda_2) = d(k\Gamma) = k^n.$$

Further, if f_1, f_2 are the quadratic forms corresponding to the dual lattices Λ_1, Λ_2, then each of f_1, f_2 is a multiple of the reciprocal of the other.

3. The Form $f_{(\lambda)}$ and its Lattice

Let V be the n-dimensional vector space over the Galois field $GF(2)$; in terms of a basis $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n$, we may write the elements as $\boldsymbol{\alpha} = \sum \alpha_i \boldsymbol{\varepsilon}_i$ with coordinates α_i which are integers taken modulo 2. The additive group of V, which we shall also denote by V, is the elementary Abelian group of order $N = 2^n$. We shall generally use group, rather than vector space, terminology; we shall, however, speak of cosets "of dimension r" or say that a given subgroup "has basis $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots$" Subgroups and cosets of dimension r will be denoted generically by V_r and C_r respectively.

In N-dimensional Euclidean space we consider integral vectors $\boldsymbol{x} = (x_{\boldsymbol{\alpha}})$ with coordinates $x_{\boldsymbol{\alpha}}$ indexed by the N elements $\boldsymbol{\alpha}$ of V. For symmetry of notation, we write $\boldsymbol{x} \cdot \boldsymbol{y}$ for the scalar product $\boldsymbol{x}'\boldsymbol{y}$.

If W is any subset of V, $[W]$ will denote the vector $\boldsymbol{x}$ defined by

$$x_{\boldsymbol{\alpha}} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \in W, \\ 0 & \text{if } \boldsymbol{\alpha} \notin W. \end{cases}$$

Let $\lambda_0, \lambda_1, \ldots, \lambda_n$ be integral exponents satisfying

(3.1) $$\lambda_0 = 0,\ \lambda_r - 1 \leqq \lambda_{r-1} \leqq \lambda_r \text{ for } 1 \leqq r \leqq n.$$

We denote by $\Lambda(\lambda) = \Lambda(\lambda_0, \lambda_1, \ldots, \lambda_n)$ the sublattice of Γ generated by all vectors $2^{\lambda_{n-r}}[C_r]$, where C_r runs over all cosets in V. Clearly

$$2^{\lambda_n}\Gamma \subset \Lambda(\lambda) \subset \Gamma.$$

We now define $f_{(\lambda)}$ to be the N-dimensional form with lattice $\Lambda(\lambda)$, so that the values assumed by $f_{(\lambda)}$ for integral values of its variables are those of

$$\boldsymbol{x}^2 = \sum_{\boldsymbol{\alpha} \in V} x_{\boldsymbol{\alpha}}^2 \text{ for } \boldsymbol{x} \in \Lambda(\lambda).$$

(We may remark here that the apparently arbitrary restrictions (3.1) involve very little loss of generality. If $\lambda_0 > 0$, we may consider $2^{-\lambda_0}\Lambda(\lambda)$, which corresponds to a multiple of $f_{(\lambda)}$. Also if the exponents satisfy only $0 \leqq \lambda_r \leqq r$ $(0 \leqq r \leqq n)$, it is not difficult to show that there exists a set (λ) defining the same lattice and satisfying (3.1), with the possible exception of the inequality $\lambda_n - 1 \leqq \lambda_{n-1}$.)

The exponents λ'_r defined by

$$\lambda'_r = \lambda_n - \lambda_{n-r} \quad (0 \leqq r \leqq n) \tag{3.2}$$

are said to be dual to the exponents λ_r. It is evident that (λ') satisfies (3.1), that $\lambda'_n = \lambda_n$, and that (λ) is dual to (λ').

We can now prove

THEOREM 3.1. (i) *$\Lambda(\lambda)$ and $\Lambda(\lambda')$ are dual lattices modulo 2^{λ_n}; $f_{(\lambda)}$ and $f_{(\lambda')}$ are multiples of reciprocal forms.*

(ii) *Let $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n$ be any basis of V. Then a basis of $\Lambda(\lambda)$ is given by the N vectors*

$$2^{\lambda_{n-r}}[V_r], \tag{3.3}$$

where V_r runs through the subgroups of V which have a subset of $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n$ as basis.

(iii) *The determinants $d(\lambda)$, $D(\lambda)$ of $\Lambda(\lambda)$, $f_{(\lambda)}$ are given by*

$$\log_2 d(\lambda) = \sum_{r=0}^{n} \lambda_r \binom{n}{r}, \tag{3.4}$$

$$\log_2 D(\lambda) = 2\sum_{r=0}^{n} \lambda_r \binom{n}{r}. \tag{3.5}$$

PROOF. (a) We first show that

$$\boldsymbol{x} \cdot \boldsymbol{y} \equiv 0 \pmod{2^{\lambda_n}} \text{ if } \boldsymbol{x} \in \Lambda(\lambda),\ \boldsymbol{y} \in \Lambda(\lambda'). \tag{3.6}$$

For this, it suffices to prove that for any cosets C_r, C'_s

$$2^{\lambda_{n-r}+\lambda'_{n-s}}[C_r] \cdot [C'_s] \equiv 0 \pmod{2^{\lambda_n}}. \tag{3.7}$$

Now if $r + s \leqq n$, then

$$\lambda_{n-r} + \lambda'_{n-s} = \lambda_n + \lambda_{n-r} - \lambda_s \geqq \lambda_n,$$

and (3.7) is trivial. If $r + s > n$, then $C_r \cap C'_s$ is either empty or a coset of dimension at least $r + s - n$; in either case

$$[C_r] \cdot [C'_s] \equiv 0 \pmod{2^{r+s-n}}.$$

Since, by (3.1), $\lambda_s - \lambda_{n-r} \leqq s - (n - r)$, we have

$$\lambda_{n-r} + \lambda'_{n-s} + r + s - n = \lambda_n + r + s - n - (\lambda_s - \lambda_{n-r}) \geqq \lambda_n,$$

and (3.7) follows at once.

(b) With the notation of part (ii) of the theorem, let $\Lambda_1(\lambda)$ be the lattice spanned by the N vectors (3.3). Then clearly

$$2^{\lambda_n}\Gamma \subset \Lambda_1(\lambda) \subset \Lambda(\lambda). \tag{3.8}$$

Further, we show that

$$\log_2 d(\Lambda_1(\lambda)) = \sum_{r=0}^{n} \lambda_r \binom{n}{r}. \tag{3.9}$$

Since there are $\binom{n}{r}$ vectors $2^{\lambda_{n-r}}[V_r]$ for each r, (3.9) will follow when we show that the set of all vectors $[V_r]$ forms a basis of Γ. To see this, suppose the $[V_r]$ ordered in such a way that the dimensions r do not decrease. Then, for each V_r, there is an $\alpha \in V$ such that $[\alpha]$ has coefficient 1 in $[V_r]$ and coefficient 0 in any predecessor of $[V_r]$; in fact, if $\varepsilon_{i_1}, \ldots, \varepsilon_{i_r}$ is a basis of V_r, $\alpha = \varepsilon_{i_1} + \ldots + \varepsilon_{i_r}$ satisfies this requirement. Hence the N unit vectors $[\alpha]$ $(\alpha \in V)$ are integral linear combinations of the $[V_r]$, whence the $[V_r]$ form a basis of Γ, as required.

(c) Let $\Lambda_1(\lambda')$ be defined as in (b) for the exponent set (λ'). By (3.9), the determinants d_1, d'_1 of $\Lambda_1(\lambda)$, $\Lambda_1(\lambda')$ satisfy

$$\begin{aligned} \log_2 (d_1 d'_1) &= \sum_{r=0}^{n} \lambda_r \binom{n}{r} + \sum_{r=0}^{n} \lambda'_{n-r} \binom{n}{n-r} \\ &= \sum \lambda_n \binom{n}{r} \\ &= \lambda_n 2^n = \log_2 (2^{N\lambda_n}) \end{aligned}$$

whence

$$d_1 d'_1 = d(2^{\lambda_n}\Gamma).$$

From this, and (3.6), it follows that $\Lambda_1(\lambda)$ and $\Lambda_1(\lambda')$ are dual lattices modulo 2^{λ_n}. But it also follows from (3.6) that $\Lambda(\lambda)$ is contained in the dual of $\Lambda_1(\lambda')$, i.e. that $\Lambda(\lambda) \subset \Lambda_1(\lambda)$. It follows from (3.8) that therefore

$$\Lambda_1(\lambda) = \Lambda(\lambda).$$

All parts of the theorem now follow at once after identifying $\Lambda(\lambda)$, $\Lambda(\lambda')$ with the dual lattices $\Lambda_1(\lambda)$, $\Lambda_1(\lambda')$.

As a corollary, we obtain

LEMMA 3.1. *$\Lambda(\lambda)$ is the set of integral $\boldsymbol{x}$ satisfying the system of congruences*

$$\sum_{\alpha \in C_r} x_\alpha \equiv 0 \pmod{2^{\lambda_r}} \tag{3.10}$$

taken over all cosets C_r of V.

PROOF. By Theorem 2.1, $\Lambda(\lambda)$ is the dual, modulo 2^{λ_n}, of $\Lambda(\lambda')$, which is generated by the vectors $2^{\lambda'_{n-r}}[C_r]$. Hence, from the definition of dual lattices, $\boldsymbol{x} \in \Lambda(\lambda)$ if and only if $\boldsymbol{x} \in \Gamma$ and

$$2^{\lambda'_{n-r}}[C_r] \cdot \boldsymbol{x} \equiv 0 \pmod{2^{\lambda_n}} \quad (0 \leqq r \leqq n).$$

Since $\lambda'_{n-r} = \lambda_n - \lambda_r$, these are precisely the congruences (3.10).

LEMMA 3.2. *$\Lambda(\lambda)$ is invariant under the following orthogonal transformations:*

(i) *the permutation of the coordinates x_α induced by the transformation*

$$\alpha \to \tau\alpha + \gamma \tag{3.11}$$

of V, where τ is a non-singular matrix over $GF(2)$ and γ is any fixed element of V;

(ii) *the involution*

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in W, \\ -x_\alpha & \text{if } \alpha \notin W, \end{cases} \tag{3.12}$$

where W is any fixed subgroup of V of dimension $n - 1$.

PROOF. (i) The transformation (3.11) of V permutes the cosets of each dimension r, and so induces a permutation of the generators $2^{\lambda_{n-r}}[C_r]$ of $\Lambda(\lambda)$.

(ii) Suppose first that $\boldsymbol{x} = 2^{\lambda_{n-r}}[C_r]$ for some coset C_r. Then $C_r \cap W$ is either empty or a coset of dimension at least $r - 1$, and in each case it is easy to show that $\boldsymbol{y}$, defined by (3.12), is a point of $\Lambda(\lambda)$. For if $C_r \cap W$ is empty, $\boldsymbol{y} = -\boldsymbol{x}$; if $C_r \cap W = C_r$, $\boldsymbol{y} = \boldsymbol{x}$; and if $C_r \cap W = C_{r-1}$ and say $C_r = C_{r-1} \cup C'_{r-1}$,

$$\begin{aligned} \boldsymbol{y} &= 2^{\lambda_{n-r}}[C_{r-1}] - 2^{\lambda_{n-r}}[C'_{r-1}] \\ &= 2^{\lambda_{n-r}+1}[C_{r-1}] - 2^{\lambda_{n-r}}[C_r] \\ &\in \Lambda(\lambda) \end{aligned}$$

since $\lambda_{n-r} + 1 \geqq \lambda_{n-r+1}$.

Since $\Lambda(\lambda)$ is generated by the vectors $2^{\lambda_{n-r}}[C_r]$, it follows that $\boldsymbol{y} \in \Lambda(\lambda)$ whenever $\boldsymbol{x} \in \Lambda(\lambda)$. Since the transformation (3.12) is involutory, the converse statement holds. $\Lambda(\lambda)$ is therefore invariant, as asserted.

Our final task in this section is to determine the minimum M of $f_{(\lambda)}$, i.e. the minimum of $\boldsymbol{x}^2$ for points $\boldsymbol{x} \neq \boldsymbol{0}$ of $\Lambda(\lambda)$. We shall show that in fact

M is the minimum of $\boldsymbol{x}^2$ over the set of vectors $2^{\lambda_{n-r}}[C_r]$ which we have selected to generate $\Lambda(\lambda)$.

For this purpose, it is convenient to define the special lattices Λ_s $(0 \leqq s \leqq n)$: Λ_s is the lattice $\Lambda(\lambda)$ whose exponents are defined by

$$\lambda_r = 0 \text{ if } r \leqq s;\ \lambda_r = 1 \text{ if } r > s. \tag{3.13}$$

LEMMA 3.3. *Suppose that* $\boldsymbol{x} \in \Lambda_s$ *and that not all* x_α *are even. Then at least* 2^{n-s} *coordinates* x_α *are odd.*

PROOF. If $s = n$, the result is trivial; hence we may suppose that $0 \leqq s < n$. We shall proceed by induction on n, the result being obvious when $n = 1$.

After applying a suitable transformation (3.11), we may suppose that x_0 is odd. By (3.10), with $r = n$, $\lambda_n = 1$, we have

$$\sum_{\alpha \in V} x_\alpha \equiv 0 \pmod 2,$$

so that some other coordinate x_γ, say, is odd. Choose an $(n-1)$-dimensional subgroup W of V so that $\gamma \notin W$, and let W' be the other coset of W.

Now since (3.10) holds a fortiori whenever $C_r \subset W$, induction on n shows that x_α is odd for at least $2^{(n-1)-s}$ indices α in W. The same result also holds for W'; for, by the transformation $\alpha \to \alpha + \gamma$ of (3.11), W' is transformed into W; and, under the induced permutation of the coordinates, x_γ, which is odd, is transformed into x_0. Thus the odd coordinates x_α number at least $2^{n-1-s} + 2^{n-1-s} = 2^{n-s}$, as asserted.

Let us now define the *rank* of a point $\boldsymbol{x} \neq \boldsymbol{0}$ of $\Lambda(\lambda)$ to be the largest r $(0 \leqq r \leqq n)$ for which all coordinates x_α are divisible by 2^{λ_r}. We then have

THEOREM 3.2. *The minimum* M *of* $f_{(\lambda)}$ *is given by*

$$\log_2 M = m = \min_r\, (n - r + 2\lambda_r). \tag{3.14}$$

A point $\boldsymbol{x} \neq \boldsymbol{0}$ *of* $\Lambda(\lambda)$ *is a minimal vector* $f_{(\lambda)}$ *if and only if it is of rank* R, *where*

$$n - R + 2\lambda_R = m, \tag{3.15}$$

and, for some subset H *of* V *containing* 2^{n-R} *elements,*

$$|x_\alpha| = 2^{\lambda_R} \text{ if } \alpha \in H,\ x_\alpha = 0 \text{ if } \alpha \notin H. \tag{3.16}$$

PROOF. Each generator $\boldsymbol{x} = 2^{\lambda_r}[C_{n-r}]$ of $\Lambda(\lambda)$ satisfies

$$\boldsymbol{x}^2 = 2^{n-r+2\lambda_r},$$

so that certainly $M \leqq 2^m$, where m is defined by (3.14).

On the other hand, let $\boldsymbol{x} \in \Lambda(\lambda)$, $\boldsymbol{x} \neq \boldsymbol{0}$, and suppose that $\boldsymbol{x}$ has rank $r (0 \leqq r \leqq n)$; set $\boldsymbol{y} = 2^{-\lambda_r}\boldsymbol{x}$, so that $\boldsymbol{y}$ is integral, $\boldsymbol{y} \neq \boldsymbol{0}$.

If now all y_α are even, then, by the definition of rank, we must have

$r = n$; since $y \neq \mathbf{0}$, we therefore have

$$x^2 = 2^{2\lambda_n} y^2 \geqq 4 \cdot 2^{2\lambda_n} > 2^m.$$

If however some y_α is odd, we see that $y \in \Lambda_r$. For, since

$$x \in \Lambda(\lambda_0, \lambda_1, \ldots, \lambda_n),\ y = 2^{-\lambda_r} x \in \Lambda(0, \ldots, 0, \lambda_{r+1} - \lambda_r, \ldots, \lambda_n - \lambda_r);$$

we have $\lambda_{r+1} > \lambda_r$, by the definition of rank, and so $\lambda_s - \lambda_r \geqq 1$ for $s > r$; hence a fortiori $y \in \Lambda(0, \ldots, 0, 1, \ldots, 1) = \Lambda_r$. Now Lemma 3.3 shows that at least 2^{n-r} coordinates y_α are odd, whence

$$x^2 = 2^{2\lambda_r} y^2 \geqq 2^{n-r+2\lambda_r} \geqq 2^m.$$

This establishes (3.14). The argument shows that in fact $x^2 = M = 2^m$ precisely when x has rank R satisfying (3.15) and the corresponding $y = 2^{-\lambda_R} x$ has 2^{n-R} coordinates ± 1 and the rest zero. The proof of the theorem is therefore complete.

4. The Extreme Forms $f_{(\lambda)}$

Although Theorem 3.2 takes us some way towards a specification of the minimal vectors of $f_{(\lambda)}$, the complete picture is rather complicated. (We shall give more precise results in § 5.) However, we can easily write down a sufficiently large set of minimal vectors to enable us to establish the extreme forms $f_{(\lambda)}$.

We denote generically by R an index satisfying (3.15), so that there are certainly minimal vectors of rank R.

Let $\mathfrak{M}_R$ denote the set of vectors

$$(4.1) \qquad 2^{\lambda_R}[C_{n-R}],\quad 2^{\lambda_R}[C_{n-R-1}] - 2^{\lambda_R}[C'_{n-R-1}]$$

and their negatives, taken over all cosets of the indicated dimensions, where C_{n-R-1}, C'_{n-R-1} denote distinct cosets of the same subgroup.

Lemma 4.1. (i) *$\mathfrak{M}_R$ is a set of $2^{n+1}K_{n,R}$ minimal vectors of $f_{(\lambda)}$ of rank R, where*

$$K_{n,R} = \frac{(2^n - 1)(2^{n-1} - 1) \ldots (2^{n-R+1} - 1)}{(2^R - 1)(2^{R-1} - 1) \ldots (2 - 1)}.$$

(ii) The group $\mathfrak{G}$ of automorphs of $f_{(\lambda)}$ is transitive on $\mathfrak{M}_R$.

Proof. Let $\mathfrak{G}'$ be the group generated by all the orthogonal transformations given in Lemma 3.2. Since these leave $\Lambda(\lambda)$ invariant, $\mathfrak{G}'$ is a subgroup of $\mathfrak{G}$.

It is now easy to see that $\mathfrak{M}_R$ is precisely the set of vectors which are the transforms by $\mathfrak{G}'$ of any one of them. For, by suitable choice of τ and γ in (3.11), any coset may be mapped into any other coset of the same dimension; and, for fixed C_{n-R-1}, C'_{n-R-1} with $C_{n-R-1} \cup C'_{n-R-1} = C_{n-R}$, the transformation (3.12) interchanges the two vectors (4.1) if W is suitably chosen.

Since $2^{\lambda_R}[C_{n-R}]$ is a generator of $\Lambda(\lambda)$, all vectors of $\mathfrak{M}_R$ belong to $\Lambda(\lambda)$; and, by the criterion of Theorem 3.2, they are all minimal vectors of $f_{(\lambda)}$. The argument also establishes part (ii) of the lemma.

Finally, if V_{n-R} is any fixed subgroup, we have from (4.1) the 2 vectors $\pm 2^{\lambda_R}[V_{n-R}]$ and the $2(2^{n-R}-1)$ vectors $2^{\lambda_R}[C_{n-R-1}] - 2^{\lambda_R}[C'_{n-R-1}]$ obtained by splitting V_{n-R} in all ways into 2 cosets. This gives 2^{n-R+1} vectors of $\mathfrak{M}_R$ corresponding to each V_{n-R}. Since V_{n-R} has 2^R distinct cosets, and V contains $K_{n,R}$ subgroups V_{n-R}, the total number of vectors in $\mathfrak{M}_R$ is $2^{n+1}K_{n,R}$, as asserted.

LEMMA 4.2. *If $0 < R < n$, $f_{(\lambda)}$ is perfect with respect to the set $\mathfrak{M}_R$ of minimal vectors; i.e. if $g(\boldsymbol{x})$ is any quadratic form satisfying*

$$g(\boldsymbol{x}) = 0 \text{ for all } \boldsymbol{x} \in \mathfrak{M}_R, \tag{4.2}$$

then $g(\boldsymbol{x}) \equiv 0$.

PROOF. Suppose that (4.2) holds, where $g(\boldsymbol{x}) = \sum_{\alpha,\beta\in V} b_{\alpha\beta}x_\alpha x_\beta$ $(b_{\alpha\beta}=b_{\beta\alpha})$; we have to show that $b_{\alpha\beta} = 0$ for all $\alpha, \beta \in V$.

Inserting the vectors (4.1), with $C_{n-R} = C_{n-R-1} \cup C'_{n-R-1}$, we obtain

$$\sum_{\alpha,\beta\in C_{n-R}} b_{\alpha\beta} = 0, \tag{4.3}$$

$$\sum_{\alpha,\beta\in C_{n-R-1}} b_{\alpha\beta} + \sum_{\alpha,\beta\in C'_{n-R-1}} b_{\alpha\beta} - 2\sum_{\substack{\alpha\in C_{n-R-1}\\ \beta\in C'_{n-R-1}}} b_{\alpha\beta} = 0,$$

whence, by addition,

$$\sum_{\alpha,\beta\in C_{n-R-1}} b_{\alpha\beta} + \sum_{\alpha,\beta\in C'_{n-R-1}} b_{\alpha\beta} = 0. \tag{4.4}$$

Applying (4.4) to each pair of 3 distinct cosets of a V_{n-R-1} (which is possible since $n - R - 1 \leqq n - 2$), we deduce that

$$\sum_{\alpha,\beta\in C_{n-R-1}} b_{\alpha\beta} = 0.$$

We now make the inductive assumption that the relations

$$\sum_{\alpha,\beta\in C_{r+1}} b_{\alpha\beta} = 0, \tag{4.5}$$

$$\sum_{\alpha,\beta\in C_r} b_{\alpha\beta} = 0 \tag{4.6}$$

hold, for some r with $0 < r \leqq n - 2$, for any cosets C_{r+1}, C_r, and prove that (4.6) holds for cosets of dimension $r - 1$.

Let V_{r-1} be a subgroup and $C_{r-1} = V_{r-1} + \gamma_1$ any coset of it. Since $r - 1 \leqq n - 3$, V_{r-1} has at least 2^3 cosets; let $C^i = V_{r+1} + \gamma_i$ $(i = 1, \ldots, 6)$

be 6 distinct cosets of V_{r-1} such that $C^i \cup C^{i+1}$ ($i = 1, 3, 5$) are 3 cosets of a subgroup V_r. We write (temporarily)

$$B_{ij} = \sum_{\substack{\alpha \in C^i \\ \beta \in C^j}} b_{\alpha\beta}.$$

From (4.6), with $C_r = C^i \cup C^j$, we have

$$B_{ii} + B_{jj} + 2B_{ij} = 0 \quad (1 \leqq i < j \leqq 6). \tag{4.7}$$

From (4.5), with $C_{r+1} = \cup_{i=1}^{4} C^i$, we have

$$\sum_{i=1}^{4} B_{ii} + 2\sum_{1 \leqq i < j \leqq 4} B_{ij} = 0. \tag{4.8}$$

Adding (4.7) for all i, j with $1 \leqq i < j \leqq 4$ and subtracting (4.8), we obtain

$$\sum_{i=1}^{4} B_{ii} = 0. \tag{4.9}$$

Adding (4.7) for $i, j = 1, 2$ and $i, j = 3, 4$, and subtracting (4.9), we obtain

$$B_{12} + B_{34} = 0.$$

From this, and the two similar relations $B_{12} + B_{56} = 0$, $B_{34} + B_{56} = 0$, we deduce that $B_{12} = 0$; hence, by (4.7),

$$B_{11} + B_{22} = 0.$$

From this and the two similar relations $B_{11} + B_{33} = 0$, $B_{22} + B_{33} = 0$, we deduce that $B_{11} = 0$, i.e.

$$\sum_{\alpha, \beta \in C_{r-1}} b_{\alpha\beta} = 0,$$

as required.

Now we have shown that (4.5), (4.6) hold for $r = n - R - 1$, where $0 \leqq n - R - 1 \leqq n - 2$. Hence by induction, (4.5) and (4.6) hold for $r = 0$. Thus, for any distinct α, β,

$$b_{\alpha\alpha} + b_{\beta\beta} + 2b_{\alpha\beta} = 0, \; b_{\alpha\alpha} = 0.$$

It follows that $b_{\alpha\beta} = 0$ for all α, β, and our proof is complete.

It is now easy to prove our main result:

THEOREM 4.1. *$f_{(\lambda)}$ is extreme if and only if it has minimal vectors of rank R for some R with $0 < R < n$, or is the 4-variable form $f_{(0,1,1)}$.*

PROOF. (i) If R satisfies (3.15), with $0 < R < n$, we have exhibited a set $\mathfrak{M}_R$ of minimal vectors such that (a) the group $\mathfrak{G}$ of automorphs of $f_{(\lambda)}$ is transitive on $\mathfrak{M}_R$ (Lemma 4.1); and (b) $f_{(\lambda)}$ is perfect with respect to $\mathfrak{M}_R$. Hence, by [1], theorem 4, $f_{(\lambda)}$ is extreme.

(ii) If (3.15) holds only with $R = 0$ or $R = n$, it may be shown that, when $f_{(\lambda)} \neq f_{(0,1,1)}$, the total number of minimal vectors of rank 0 or n is

$2^{n+1} = 2^{n+1}K_{n,0}$. Thus $f_{(\lambda)}$ has at most $2^{n+1} = 2N$ pairs of minimal vectors. Since $2N < \frac{1}{2}N(N+1)$ if $N \geqq 4$, $f_{(\lambda)}$ is not perfect, and so not extreme, if $N \geqq 4$; and trivially $f_{(\lambda)}$ is not perfect if $N = 2$.

Although the 2^n choices of (λ) satisfying (3.1) yield 2^n distinct forms $f_{(\lambda)}$ for each n, most of which are extreme, these forms are not all inequivalent. The following two theorems appear to settle the problem of finding the inequivalent $f_{(\lambda)}$, at least for small N.

THEOREM 4.2. *For any set* (λ) *of exponents satisfying* (3.1), *define the conjugate set* (μ) *by*

$$\mu_r = r + \lambda_{n-r} - \lambda_n. \tag{4.10}$$

Then (μ) *satisfies* (3.1) *and*

$$2^{-\mu_n} f_{(\mu)} \sim 2^{-\lambda_n} f_{(\lambda)}. \tag{4.11}$$

PROOF. From (4.10),

$$\mu_0 = 0,\ \mu_r - \mu_{r-1} = 1 - (\lambda_{n-r+1} - \lambda_{n-r}) = 0 \text{ or } 1 \quad (1 \leqq r \leqq n)$$

so that (μ) satisfies (3.1). It is also clear that (λ) is conjugate to (μ), i.e. $\lambda_r = r + \mu_{n-r} - \mu_n$.

Now let B: $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be any fixed basis of V, and define a scalar product on V by

$$\alpha \cdot \beta = \sum_{i=1}^{n} \alpha_i \beta_i \text{ if } \alpha = \sum \alpha_i \varepsilon_i,\ \beta = \sum \beta_i \varepsilon_i .$$

$\alpha \cdot \beta$ is thus an element of $GF(2)$, written as an integer modulo 2. We now consider the transformation

$$y_\alpha = 2^{-\lambda_n} \sum_{\beta \in V} (-1)^{\alpha \cdot \beta} x_\beta, \tag{4.12}$$

and we shall show that

$$y \in \Lambda(\mu) \text{ if } x \in \Lambda(\lambda), \tag{4.13}$$

$$2^{-\mu_n} y^2 = 2^{-\lambda_n} x^2. \tag{4.14}$$

By Theorem 3.1 (ii), a basis of $\Lambda(\lambda)$ is given by the vectors $2^{\lambda_{n-r}}[V_r]$, where V_r runs through the subgroups of V having a subset of B as basis. Hence, to prove (4.13), it suffices to show that $y \in \Lambda(\mu)$ if $x = 2^{\lambda_{n-r}}[V_r]$ $(0 \leqq r \leqq n)$. For each such V_r, let V'_{n-r} be the complementary subgroup of V; i.e. V_r and V'_{n-r} have complementary subsets of B as basis. We then have

$$\sum_{\beta \in V_r} (-1)^{\alpha \cdot \beta} = \begin{cases} 2^r \text{ if } \alpha \in V'_{n-r}, \\ 0 \text{ if } \alpha \notin V'_{n-r}. \end{cases} \tag{4.15}$$

For if $\alpha \in V'_{n-r}$, then $\alpha \cdot \beta = 0$ for all $\beta \in V_r$. If, however, $\alpha \notin V'_{n-r}$, then there is some $\gamma \in V_r$ with $\alpha \cdot \gamma = 1$; then

$$\sum_{\beta \in V_r} (-1)^{\alpha\cdot\beta} = \sum_{\beta \in V_r} (-1)^{\alpha\cdot(\beta+\gamma)} = -\sum_{\beta \in V_r} (-1)^{\alpha\cdot\beta} = 0.$$

If now $x = 2^{\lambda_{n-r}}[V_r]$, (4.12) and (4.15) give

$$y_\alpha = 2^{-\lambda_n} \sum_{\beta \in V_r} (-1)^{\alpha\cdot\beta} 2^{\lambda_{n-r}} = \begin{cases} 2^{r+\lambda_{n-r}-\lambda_n} & \text{if } \alpha \in V'_{n-r}, \\ 0 & \text{if } \alpha \notin V'_{n-r}, \end{cases}$$

i.e. $y = 2^{\mu_r}[V'_{n-r}]$, which is a generator of $\Lambda(\mu)$. This proves (4.13).

To prove (4.14), we use the case $r = n$, $V_r = V$ of (4.15) to obtain

$$y^2 = \sum_{\alpha \in V} y_\alpha^2 = 2^{-2\lambda_n} \sum_{\alpha, \beta, \gamma \in V} (-1)^{\alpha\cdot(\beta+\gamma)} x_\beta x_\gamma = 2^{n-2\lambda_n} \sum_{\alpha \in V} x_\alpha^2,$$

whence (4.14) follows at once.

The desired equivalence (4.11) follows from (4.13), (4.14), on observing that, since the relations between (λ), (μ) and between x, y are symmetrical, $y \in \Lambda(\mu)$ if and only if $x \in \Lambda(\lambda)$.

THEOREM 4.3. *For the exponent sets* (λ), (μ) *given by*

$$\lambda_r = \left[\frac{r}{2}\right] \qquad (0 \leqq r \leqq n), \tag{4.16}$$

$$\mu_r = \left[\frac{r+1}{2}\right] \qquad (0 \leqq r \leqq n), \tag{4.17}$$

we have

$$f_{(\mu)} \sim 2f_{(\lambda)}$$

PROOF. Take any fixed subgroup W of dimension $n-1$ and any element $\gamma \notin W$, and consider the transformation defined by

$$\begin{aligned} y_\alpha &= x_\alpha + x_{\alpha+\gamma} \\ y_{\alpha+\gamma} &= x_\alpha - x_{\alpha+\gamma} \end{aligned} \quad (\alpha \in W) \tag{4.18}$$

Then clearly $y^2 = 2x^2$, and so the required equivalence will follow when we show that $y \in \Lambda(\mu)$ if and only if $x \in \Lambda(\lambda)$.

Let V_r be any subgroup of V and $x = 2^{\lambda_{n-r}}[V_r] \in \Lambda(\lambda)$. Then $V_r \cap W$ is either V_r or a subgroup V_{r-1}; we must now distinguish three cases.

(a) If $V_r \subset W$, (4.18) shows that $y = 2^{\lambda_{n-r}}[V_r] + 2^{\lambda_{n-r}}[V_r + \gamma]$, i.e. $y = 2^{\lambda_{n-r}}[V_{r+1}] = 2^{\mu_{n-r-1}}[V_{r+1}] \in \Lambda(\mu)$.

(b) If $V_r \cap W = V_{r-1}$ and $V_r = V_{r-1} \cup (V_{r-1} + \gamma)$, (4.18) gives

$$y = 2.2^{\lambda_{n-r}}[V_{r-1}] = 2^{\mu_{n-r+1}}[V_{r-1}] \in \Lambda(\mu).$$

(c) The remaining possibility is that $V_r = V_{r-1} \cup (V_{r-1} + \beta)$, where $V_{r-1} \subset W$, $V_{r-1} + \beta \subset W + \gamma$, but the cosets $V_{r-1} + \beta$, $V_{r-1} + \gamma$ are distinct. Then observing that $V_{r-1} + \beta + \gamma \subset W$, we obtain from (4.18)

$$y = 2^{\lambda_{n-r}}\{[V_{r-1}] + [V_{r-1} + \gamma] + [V_{r-1} + \beta + \gamma] - [V_{r-1} + \beta]\}.$$

With $V_{r+1} = V_{r-1} \cup (V_{r-1} + \beta) \cup (V_{r-1} + \gamma) \cup (V_{r-1} + \beta + \gamma)$, this gives

$$\begin{aligned} y &= 2^{\lambda_{n-r}}[V_{r+1}] - 2.2^{\lambda_{n-r}}[V_{r-1} + \beta] \\ &= 2^{\mu_{n-r-1}}[V_{r+1}] - 2^{\mu_{n-r+1}}[V_{r-1} + \beta] \in \Lambda(\mu). \end{aligned}$$

We have therefore shown that, in all cases, $y \in \Lambda(\mu)$ if $x = 2^{\lambda_{n-r}}[V_r]$; since these vectors generate $\Lambda(\lambda)$, it follows that $y \in \Lambda(\mu)$ if $x \in \Lambda(\lambda)$. A precisely similar argument, using the inverse transformation $x_\alpha = \frac{1}{2}(y_\alpha + y_{\alpha+\gamma})$ $x_{\alpha+\gamma} = \frac{1}{2}(y_\alpha - y_{\alpha+\gamma})$, shows that conversely $x \in \Lambda(\lambda)$ if $y \in \Lambda(\mu)$. This completes the proof of the theorem.

The particular interest of the (equivalent) exponent sets (4.16) and (4.17) is shown by:

THEOREM 4.4. *For each* $N = 2^n (n \geqq 2)$, *the extreme form* f_N *whose exponents are given by* (4.16) *has*

$$\gamma_N(f_N) = (\tfrac{1}{2}N)^{\frac{1}{2}}; \tag{4.19}$$

and this is the largest value of $\gamma_N(f_{(\lambda)})$.

PROOF. Since $\gamma_N(f) = M/D^{1/N}$, the values of M and D given in Theorems 3.1 and 3.2 show that

$$\begin{aligned} \log_2 \gamma_N(f_{(\lambda)}) &= m - \frac{2}{N} \sum_{r=0}^{n} \lambda_r \binom{n}{r} \\ &= \tfrac{1}{2}n - \frac{1}{N} \sum_{r=0}^{n} (n - r + 2\lambda_r - m) \binom{n}{r}. \end{aligned}$$

Since $m = \min (n - r + 2\lambda_r)$, we have $n - r + 2\lambda_r - m \geqq 0$ for all r; and the parity of $n - r + 2\lambda_r - m$ is determined by the parity of r. It is thus easy to see that $\gamma_N(f_{(\lambda)})$ is greatest when the expressions $n-r+2\lambda_r-m$ $(r = 0, 1, \ldots, n)$ take alternately the values 0 and 1; and the only exponent sets satisfying this condition are those given in (4.16) and (4.17). This shows that $\gamma_N(f_N)$ is maximal, and a simple calculation now gives (4.19).

It is perhaps worth noting here that, by Theorem 3.1 (i), the reciprocal of $f_{(\lambda)}$ is a multiple of $f_{(\lambda')}$, where $\lambda'_r = \lambda_n - \lambda_{n-r}$ $(0 \leqq r \leqq n)$. For the exponent set (4.16) corresponding to f_N, it is easily verified that the dual set (λ') is either (4.16) or (4.17), according as n is odd or even; thus f_N *is equivalent to (a multiple of) its reciprocal.*

5. The Minimal Vectors of $f_{(\lambda)}$

For each R satisfying (3.15), we have exhibited a set $\mathfrak{M}_R$ of $2^{n+1}K_{n,R}$ minimal vectors of $f_{(\lambda)}$. Denoting by s_R the total number of pairs of minimal vectors of rank R, we therefore have certainly

$$2s_R \geqq 2^{n+1}K_{n,R}. \tag{5.1}$$

It is not difficult to obtain an upper bound for s_R, in the following way. First, the second part of theorem 3.2 may be sharpened to the statement that $\boldsymbol{x} \in \Lambda(\lambda)$ is a minimal vector of rank R if and only if, for some *coset* C_{n-R}, we have

$$(5.2) \qquad |x_\alpha| = 2^{\lambda_R} \text{ if } \alpha \in C_{n-R},\ x_\alpha = 0 \text{ otherwise.}$$

We may say that a minimal vector (5.2) has carrier C_{n-R}. By using the transformation (3.11), we see that the number of minimal vectors with carrier C_{n-R} is the same for all cosets of dimension $n - R$; call this number N_R, so that

$$(5.3) \qquad 2s_R = 2^R K_{n,R} N_R.$$

For a minimal vector (5.2) with carrier a subgroup V_{n-R}, set

$$(5.4) \qquad x_\alpha = 2^{\lambda_R}(1 - 2z_\alpha) \text{ if } \alpha \in V_{n-R}, \quad x_\alpha = 0 \text{ otherwise,}$$

so that z_α is defined on V_{n-R} and has the value 0 or 1. It may now be verified that $\boldsymbol{x} \in \Lambda(\lambda)$ if and only if

$$(5.5) \quad \boldsymbol{z} \in \Lambda(0,\ 0,\ \lambda_{R+2} - \lambda_R - 1,\ \lambda_{R+3} - \lambda_R - 1, \ldots, \lambda_n - \lambda_R - 1)$$

(a 2^{n-R} -dimensional lattice), with each $z_\alpha = 0$ or 1.

We must now distinguish the cases: $\lambda_{R+2} = \lambda_R + 2$; $\lambda_{R+2} = \lambda_R + 1$.

(a) If $\lambda_{R+2} = \lambda_R + 2$, (5.5) implies that certainly

$$(5.6) \qquad \boldsymbol{z} \in \Lambda(0,\ 0,\ 1,\ 1, \ldots, 1),\ z_\alpha = 0 \text{ or } 1;$$

the number of solutions of (5.6) is precisely the number of solutions in $GF(2)$ of a set of equations of rank $\binom{n-R}{2} + \ldots + \binom{n-R}{n-R}$, and nullity $1 + \binom{n-R}{1}$, i.e. it is 2^{1+n-R}. Thus now

$$N_R \leqq 2^{1+n-R}\ , \ 2s_R \leqq 2^{n+1} K_{n,R}.$$

This shows that the bound (5.1) is precise, i.e. that

$$2s_R = 2^{n+1} K_{n,R} \text{ if } \lambda_{R+2} = \lambda_R + 2.$$

The same result is easily seen to hold if λ_{R+2} is undefined, i.e. if $R \geqq n-1$.

(b) If $\lambda_{R+2} = \lambda_R + 1$, then $\lambda_{R+3} = \lambda_{R+2} + 1$ and (5.5) implies that

$$(5.7) \qquad \boldsymbol{z} \in \Lambda(0,\ 0,\ 0,\ 1,\ 1, \ldots, 1), \quad z_\alpha = 0 \text{ or } 1.$$

Arguing as above, we obtain

$$(5.8) \qquad N_R \leqq 2^{1+\binom{n-R}{1}+\binom{n-R}{2}},$$

whence by (5.3)

$$(5.9) \qquad 2s_R \leqq 2^{n+1+\binom{n-R}{2}} K_{n,R}.$$

By a deeper investigation into the case $\lambda_{R+2} = \lambda_R + 1$, based on the theory of $(n - R)$-dimensional quadratic forms over $GF(2)$, we have established the precise result

$$(5.10)\quad 2s_R = 2^{n+1}K_{n,R} \sum_{0 \leqq \delta \leqq d} 2^{\delta(\delta-1)} \frac{(2^{n-R}-1)(2^{n-R-1}-1)\dots(2^{n-R-2\delta+1}-1)}{(4^{\delta}-1)(4^{\delta-1}-1)\dots(4-1)}$$

where $d \geqq 0$ is the largest integer such that there are minimal vectors of ranks $R,\ R+2,\ R+4, \dots,\ R+2d$; alternatively expressed, d is the largest integer for which

$$(5.11)\qquad \lambda_{R+i} = \lambda_R + \left[\frac{i+1}{2}\right] \text{ for } 0 \leqq i \leqq 2d.$$

From (5.10), or pursuing the direct argument which led to (5.9), it follows that the bound (5.9) is precise if (5.11) holds for all $i \geqq 0$. In particular, for the form f_N whose exponents are given by (4.16) (or equivalently by (4.17)) we obtain

$$2s_R = 2^{n+1+\binom{n-R}{2}} K_{n,R} \text{ for all odd } R \leqq n,$$

whence

$$s(f_N) = 2^n \sum_{R \text{ odd}} 2^{\binom{n-R}{2}} K_{n,R}.$$

6. Conclusion

Our analysis of the lattices $\Lambda(\lambda)$ has yielded a large number of extreme forms $f_{(\lambda)}$, nearly all of which are new. The form $f_{(0,\dots,0,1)}$ is the known form B_N of [2]. The special form f_N, corresponding to the exponent set $\lambda_r = [\frac{1}{2}r]$ $(0 \leqq r \leqq n)$, is equivalent to the known absolutely extreme form when $N = 4$ or $N = 8$, and may well be absolutely extreme for some larger N.

For all sufficiently large N, f_N cannot be absolutely extreme, since $\gamma_N(f_N)$ is of order $N^{\frac{1}{2}}$ only. We can hope that, by methods similar to those used here, a sequence of forms can be constructed with larger values of $\gamma_N(f)$ for large N. A preliminary investigation suggests the existence of such a sequence with $\gamma_N(f)$ of order $N^{2/3}$ whenever $N = 2.3^n$.

We add finally some notes on further results which may be obtained from our analysis of the lattices $\Lambda(\lambda)$.

(i) By choosing suitable sublattices of $\Lambda(\lambda)$ of lower dimension, it is possible to construct further forms with relatively large values of $\gamma_N(f)$. Thus the sublattice of $\Lambda(0,\ 1,\ 1,\ 2)$ defined by

$$\sum_{\alpha \in V_3} x_\alpha = 0$$

gives a 7-variable form with $\gamma_7(f) = 2^{6/7}$; since $\gamma_7 = 2^{6/7}$, this form is absolutely extreme. Similarly, the sublattice of $\Lambda(0,\ 0,\ 1,\ 1,\ 2)$ defined by

$$\sum_{\alpha \in V_4} x_\alpha = 0$$

gives a 15-variable form with $\gamma_{15}(f) = 2^{7/5}$; thus we obtain the new inequalities

$$\gamma_{15} \geqq 2^{7/5};\ \Delta_{15} \leqq 2^{1/6}.$$

(ii) Many of our results will apply with very little modification to give upper bounds for the critical determinant of the N-dimensional convex body

$$K_\nu\colon \sum_{\alpha \in V} |x_\alpha|^\nu \leqq 1 \quad (\nu \geqq 1,\ V = V_n).$$

For example, for $\boldsymbol{x} \in \Lambda(0, 1, 2, \ldots, n)$, $\boldsymbol{x} \neq \boldsymbol{0}$, we find that

$$\min \sum_{\alpha \in V} |x_\alpha| = 2^n = N;$$

thus, for the 'octahedron' K_1, we have

$$\Delta(K_1) \leqq d(N^{-1}\Lambda(0, 1, 2, \ldots, n)) = N^{-N} 2^{\Sigma r \binom{n}{r}} = N^{-\frac{1}{2}N} \quad (N = 2^n).$$

This represents an improvement on known results for small $N \geqq 4$.

We append a table of the distinct extreme forms $f_{(\lambda)}$ for $N = 4, 8, 16$ and 32, giving the values of $\log_2 M = m$; $\log_2 D$; the number s of pairs of minimal vectors; and $\log_2 \Delta$ (where $\Delta = (2/M)^N D$). The values of Δ for $f_{(0,0,1)}$ and $f_{(0,0,1,1)}$ show that they are the known absolutely extreme forms in 4 and 8 variables respectively. The italicized figures in the exponent sets (λ) are the λ_R for which there exist minimal vectors of rank R, i.e. for which $n - R + 2\lambda_R = m$.

N	(λ)	$\log_2 M$	$\log_2 D$	s	$\log_2 \Delta$
4	(0, *0*, 1)	1	2	12	2
8	(0, 0, *0*, 1)	1	2	56	2
	(0, *0*, 1, *1*)	2	8	120	0
16	(0, 0, 0, *0*, 1)	1	2	240	2
	(0, 0, *0*, 1, *1*)	2	10	1,136	−6
	(0, 0, *0*, 1, 2)	2	12	560	−4
	(0, *0*, 1, *1*, 2)	3	24	2,160	−8
	(0, *0*, 1, 2, 2)	3	32	240	0
32	(0, 0, 0, 0, *0*, 1)	1	2	992	2
	(0, 0, 0, *0*, 1, *1*)	2	12	9,952	−20
	(0, 0, 0, *0*, 1, 2)	2	14	4,960	−18
	(0, 0, *0*, 1, *1*, 2)	3	34	40,672	−30
	(0, 0, *0*, 1, 2, 2)	3	44	4,960	−20
	(0, 0, 1, 1, *1*, 2)	3	54	992	−10
	(0, *0*, 1, *1*, 2, *2*)	4	64	73,440	−32
	(0, *0*, 1, *2*, 2, *2*)	4	84	1,024	−12
	(0, *0*, 1, 2, 3, 3)	4	96	992	0
	(0, 1, 1, *1*, 2, *2*)	4	74	9,952	−22

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The University of Sydney.