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BELIEVING THE AXIOMS. I

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Nothing venture, nothing win,
Blood is thick, but water's thin.
—Gilbert & Sullivan

§0. Introduction. Ask a beginning philosophy of mathematics student why we believe the theorems of mathematics and you are likely to hear, "because we have proofs!" The more sophisticated might add that those proofs are based on true axioms, and that our rules of inference preserve truth. The next question, naturally, is why we believe the axioms, and here the response will usually be that they are "obvious", or "self-evident", that to deny them is "to contradict oneself" or "to commit a crime against the intellect". Again, the more sophisticated might prefer to say that the axioms are "laws of logic" or "implicit definitions" or "conceptual truths" or some such thing.

Unfortunately, heartwarming answers along these lines are no longer tenable (if they ever were). On the one hand, assumptions once thought to be self-evident have turned out to be debatable, like the law of the excluded middle, or outright false, like the idea that every property determines a set. Conversely, the axiomatization of set theory has led to the consideration of axiom candidates that no one finds obvious, not even their staunchest supporters. In such cases, we find the methodology has more in common with the natural scientist's hypotheses formation and testing than the caricature of the mathematician writing down a few obvious truths and preceeding to draw logical consequences.

The central problem in the philosophy of natural science is when and why the sorts of facts scientists cite as evidence really are evidence. The same is true in the case of mathematics. Historically, philosophers have given considerable attention to the question of when and why various forms of logical inference are truth-preserving. The companion question of when and why the assumption of various axioms is justified has received less attention, perhaps because versions of the "self-evidence" view live on, and perhaps because of a complacent if-thenism. For whatever reasons, there has been little attention to the understanding and

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classification of the sorts of facts mathematical scientists cite, let alone to the philosophical question of when and why those facts constitute evidence.

The question of how the unproven can be justified is especially pressing in current set theory, where the search is on for new axioms to determine the size of the continuum. This pressing problem is also the deepest that contemporary mathematics presents to the contemporary philosopher of mathematics. Not only would progress towards understanding the process of mathematical hypothesis formation and confirmation contribute to our philosphical understanding of the nature of mathematics, it might even be of help and solace to those mathematicians actively engaged in the axiom search.

Before we can begin to investigate when and why the facts these mathematicians cite constitute good evidence, we must know what facts those are. What follows is a contribution to this preliminary empirical study (thus the reference to "believing" rather than "knowing" in my title). In particular, I will concentrate on the views of the Cabal seminar, whose work centers on determinacy and large cardinal assumptions.[1] Along the way, especially in the early sections, the views of philosophers and set theorists outside the group, and even opposed to it, will be mentioned, but my ultimate goal is a portrait of the general approach that guides the Cabal's work.[2]

Because of its length, this survey appears in two parts. The first covers the axioms of ZFC, the continuum problem, small large cardinals and measurable cardinals. The second concentrates on determinacy hypothesis and large large cardinals, and concludes with some philosophical observations.

§I. The axioms of ZFC. I will start with the well-known axioms of Zermelo-Fraenkel set theory, not so much because I or the members of the Cabal have anything particularly new to say about them, but more because I want to counteract the impression that these axioms enjoy a preferred epistemological status not shared by new axiom candidates. This erroneous view is encouraged by set theory texts that begin with "derivations" of ZFC from the iterative conception, then give more self-conscious discussions of the pros and cons of further axiom candidates as they arise. The suggestion is that the axioms of ZFC follow directly from the concept of set, that they are somehow "intrinsic" to it (obvious, self-evident), while other axiom candidates are only supported by weaker, "extrinsic" (pragmatic, heuristic) justifications, stated in terms of their consequences, or intertheoretic connections, or

[1] Naturally the various members of the Cabal do not agree on everything. When appropriate, I will take note of these disagreements.

[2] I am indebted to John Burgess, for introducing me to much of the material discussed here, to Matt Foreman, Menachem Magidor, Yiannis Moschovakis, John Steel, and Hugh Woodin, for helpful conversations, to Chris Freiling, Stewart Shapiro, John Simms, and an anonymous referee for helpful comments on earlier drafts, and, especially, to Tony Martin, without whose patience and generosity the project would have been dead in the water. Versions of this work have been delivered to helpful audiences at the University of California at Los Angeles (Department of Mathematics) and Irvine (Department of Philosophy), and at conventions of the Philosophy of Science and the American Philosophical Associations. The support of NSF Grant No. SES-8509026 and the hospitality of the UCLA Math Department are also gratefully acknowledged.

explanatory power, for example. (It is these extrinsic justifications that often mimic the techniques of natural science.) Thus some mathematicians will stand by the truth of any consequence of ZFC, but dismiss additional axioms and their consequences as metaphysical rot. Even the most cursory look at the particular axioms of ZFC will reveal that the line between intrinsic and extrinsic justification, vague as it might be, does not fall neatly between ZFC and the rest. The fact that these few axioms are commonly enshrined in the opening pages of mathematics texts should be viewed as an historical accident, not a sign of their privileged epistemological or metaphysical status.

The impulse towards axiomatization can be seen as beginning in 1883, when Cantor introduced "a law of thought",

> ... fundamental, rich in consequences, and particularly marvelous for its general validity... It is always possible to bring any *well-defined* set into the *form* of a *well-ordered* set.

(Cantor [1883, p. 550], as translated in Moore [1982, p. 42]). Hallet [1984, pp. 156–157] traces Cantor's belief in the well-ordering principle to his underlying conviction that infinite sets are not so different from finite ones, that the most basic properties are ones they share. (This will be called "Cantorian finitism" in what follows. In this case, the basic property shared is "countability" or "enumerability".) Unfortunately, the mathematical community at large did not find it obvious that infinite sets could be well-ordered, and by 1895, Cantor himself came to the conclusion that his principle should really be a theorem.

Though Cantor made various efforts to prove this and related theorems (see, for example, his famous letter to Dedekind [1899]), the first proof was Zermelo's in [1904]. This proof, and especially the Axiom of Choice on which it was based, created a furor in the international mathematical community. Under the influence of Hilbert's axiomatics, Zermelo hoped to secure his proof by developing a precise list of the assumptions it required, and proposing them (in [1908]) as an axiomatic foundation for the theory of sets. The fascinating historical ins and outs of this development are clearly and readably described in Moore's book. The point of interest here is that the first axioms for set theory were motivated by a pragmatic desire to prove a particular theorem, not a foundational desire to avoid the paradoxes.[3]

For our purposes, it will be enough to give a brief survey of the arguments given by Zermelo and later writers in support of the various axioms of ZFC.

I.1. *Extensionality.* Extensionality appeared in Zermelo's list without comment, and before that in Dedekind's [1888, p. 45]. Of all the axioms, it seems the most "definitional" in character; it distinguishes sets from intensional entities like

[3]See Moore [1982]. Apparently Zermelo discovered the paradox some two years before Russell. On his interpretation, it shows only that no set can contain all its subsets as members (see Moore [1982, p. 89]). Recall that Cantor also took the paradoxes less seriously than the philosophers, for example, in the letter to Dedekind [1899]. Gödel also expresses the view that the paradoxes present a problem for logic, not for mathematics (Godel [1944/67, p. 474]).

properties or concepts. Most writers seem to echo the opinion of Boolos [1971, p. 501], that if any sense can be made of the distinction between analytic and synthetic, then the Axiom of Extensionality should be counted as analytic. (See also Shoenfield [1977, p. 325], and Wang [1974, p. 533].)

Fraenkel, Bar-Hillel and Levy give a bit more in their [1973, pp. 28, 87]. They argue that an extensional notion of set is preferable because it is simpler, clearer, and more convenient, because it is unique (as opposed to the many different ways intensional collections could be individuated), and because it can simulate intensional notions when the need arises (e.g. two distinguishable "copies" of an extensional set can be produced by taking its cross product with distinct singletons). Thus extrinsic reasons are offered even for this most obvious of axioms.

I.2. *Foundation.* Zermelo used a weak form of the Axiom of Foundation ($A \notin A$) to block Russell's paradox in a series of lectures in the summer of 1906, but by 1908, he apparently felt that the form of his Separation Axiom was enough by itself, and he left the earlier axiom off his published list. (See Moore [1982, p. 157]; Hallet [1984, p. 252].) Later Mirimanoff [1917] defined "ordinary sets" to be those without infinite descending epsilon chains. Using the notion of rank, he was able to formulate necessary and sufficient conditions for the existence of ordinary sets. Though he did not suggest that the ordinary sets are all the sets, he did think that restricting attention to them (in effect adopting Foundation) was a good working method.

This attitude towards Foundation is now a common one. It is described as weeding out "pathologies" or "oddities" (Boolos [1971, p. 491]) on the grounds that

> ... no field of set theory or mathematics is in any general need of sets which are not well-founded.

(Fraenkel, Bar-Hillel and Levy [1973, p. 88]) Von Neumann adopted it in [1925], hoping to increase the categoricity of his axioms, and Zermelo included it in [1930] because it was satisfied in all known applications of set theory and because it gives a useful understanding of the universe of sets. (Supporters of the "iterative conception" discussed below often see foundation as built into the very idea of the stages. See Boolos [1971, p. 498]; Shoenfield [1977, p. 327].)

I.3. *Pairing and Union.* Cantor first stated the Union Axiom in a letter to Dedekind in 1899 (see Moore [1982, p. 54]), and the Pairing Axiom superseded Zermelo's 1908 Axiom of Elementary Sets when he presented the modified verison of his axiom system in [1930]. Both are nearly too obvious to deserve comment from most commentators. When justifications are given, they are based on one or the other of two rules of thumb. These are vague intuitions about the nature of sets, intuitions too vague to be expressed directly as axioms, but which can be used in plausibility arguments for more precise statements. We will meet with a number of these along the way, and the question of their genesis and justification is of prime importance. For now, the two in question are *limitation of size* and *the iterative conception.*

Limitation of size came first. Hallet [1984] traces it to Cantor, who held that transfinites are subject to mathematical manipulation much as finites are (as mentioned above), while the absolute infinity (all finites and transfinites) is God and incomprehensible. Later more down-to-earth versions like Fraenkel's hold that the paradoxes are generated by postulating sets that are "too large", and that set theory will be safe if it only eschews such collections. (Hallet gives a historical and philosophical treatment of the role of this rule of thumb in the development of modern axiomatic set theory.) Thus, for example, Fraenkel, Bar-Hillel and Levy [1973] argue that a pair set is of "very modest size", and that the Union Axiom will not produce any thing "too large", because

> ... the sets whose union is to be formed will not be taken arbitrarily—they must be members of a single given set". (pp. 32–34)

(Hallet, incidentally, disagrees about Union. See [1984, pp. 209–210].)

The iterative conception originated with Zermelo [1930] (prefigured perhaps in Mirimanoff [1917]). Although Cantor, Fraenkel, Russell [1906], Jordain [1904], [1905], von Neumann [1923] and others all appealed to *limitation of size, the iterative conception* is more prevalent today. Because of its general familiarity, I shall not pause to describe it here. (See e.g. Boolos [1971] or Shoenfield [1977].) For the record, then, given two objects a and b, let A and B be the stages at which they first appear. (On the iterative picture, everything appears at some stage.) Without loss of generality, suppose B is after A. Then the pair set of a and b appears at the stage immediately following B. Similarly, if a family of sets f appears at stage F, then all members of f, and hence all members of members of f, appear before F. Thus the union of f appears at or before F. (Arguments of this form are given in Boolos [1971, p. 496] and Shoenfield [1977, p. 325].)

I.4. *Separation.* The Axiom of Separation is in many ways the most characteristic of Zermelo's axioms. Here he sees himself as giving us as much of the naive comprehension scheme as possible without inconsistency [1908, p. 202]. We see here the emergence of another rule of thumb: *one step back from disaster.* The idea here is that our principles of set generation should be as strong as possible short of contradiction. If a natural principle leads to contradiction, this rule of thumb recommends that we weaken it just enough to block the contradiction. We shall meet this principle again in [BaII, §VI.3].

Zermelo steps back in two ways. First,

> ... sets may never be *independently defined* ... but must always be *separated* as subsets of sets already given.

[1908, p. 202]. Predictably, Fraenkel, Bar-Hillel and Levy see this as the result of applying *limitation of size* to unlimited comprehension (p. 36). Zermelo's second modification is to require that the separating property be "definite" (p. 202), which he understood as ruling out such troublesome turns of natural language as "definable in a finite number of words". The vagueness of the term "definite" brought Zermelo's Separation Axiom under considerable fire until Skolem suggested that "definite" be replaced by "formula of first-order logic". (Even then, Zermelo himself held to a second-order version. See his [1930].)

Advocates of *the iterative conception* have no trouble with Separation: all the members of a are present before a, so any subset of a appears at or before the stage at which a itself appears (Boolos [1971, p. 494]; Shoenfield [1977, p. 325]).[4]

I.5. *Infinity.* The Axiom of Infinity is a simple statement of Cantor's great breakthrough. The rather colorless idea of a collection of elements that had lurked in the background of mathematical thought since prehistory might have remained there to this day if Cantor had not had the audacity to assume that they could be infinite. This was the bold and revolutionary hypothesis that launched modern mathematics; it should be seen as nothing less.

Hallet, in his historical study of Cantorian thought, enshrines Cantor's perspective into a rule of thumb called *Cantorian finitism*: infinite sets are like finite ones. (This was mentioned above in connection with Cantor's belief in the well-ordering principle.) The rule and its applications are justified in terms of their consequences. In this case:

> Dealing with natural numbers without having the set of all natural numbers does not cause more inconvenience than, say, dealing with sets without having the set of all sets. Also the arithmetic of the rational numbers can be developed in this framework. However, if one is already interested in analysis then infinite sets are indispensable since even the notion of a real number cannot be developed by means of finite sets only. Hence we have to add an existence axiom that guarantees the existence of an infinite set.

(Fraenkel, Bar-Hillel and Levy [1973, p. 45]). *Iterative conception* theorists now often take the existence of an infinite stage as part of the intuitive picture (see Boolos [1971, p. 492]; Shoenfield [1977, p. 324]), but this would hardly have come to pass if Cantor had not taken a chance and succeeded in showing that we can reason consistently about the infinite and that we have much to gain by doing so (see epigraph).

I.6. *Power set. Cantorian finiteness* yields an argument for the Power Set Axiom, as it is presumably uncontroversial that finite sets have power sets. *The iterative conception* also makes quick work. If a appears at A, then all the elements of a appear before A, so any subset of a appears at or before A. Thus the power set of a appears at the stage after A. Advocates of *limitation of size* suggest that the power set of a given set will not be large because all its members must be subsets of something small.

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[4] Wang [1974] has a more philosophical account of the iterative picture in terms of what we can "run through in intuition". Thus his justification of Separation is:

> Since x is a set, we can run through all the members of x, and, therefore, we can do so with arbitrary omissions. In particular, we can in an idealized sense check against A and delete only those members of x which are not in A. In this way, we obtain an overview of all the objects in A and recognize A as a set. (p. 533)

Parsons [1977] points out that this puts a terrible strain on the notion of intuition, and that the problem becomes worse in the case of the Power Set Axiom. See also their exchange on Replacement.

Hallet casts some well-deserved doubt on this last form of justification for the Power Set Axiom, but he does not mean to reject the axiom entirely. Instead, he resorts to a series of extrinsic justifications, the simplest of which is reminiscent of that given above by Fraenkel, Bar-Hillel and Levy for Infinity, namely, that Power Set is indispensable for a set-theoretic account of the continuum:

> This does not prove the legitimacy of the power-set principle. For the argument is not: we have a perfectly clear intuitive picture of the continuum, and the power-set principle enables us to capture this set-theoretically. Rather, the argument is: the power-set principle ... was revealed in our attempts to make our intuitive picture of the continuum analytically clearer; in so far as these attempts are successful, then the power-set principle gains some confirmatory support. (p. 213)

Not surprisingly, a similar extrinsic support for the Power Set Axiom is to be found in Fraenkel, Bar-Hillel and Levy (pp. 34–35).

I.7. *Choice.* The Axiom of Choice has easily the most tortured history of all the set-theoretic axioms; Moore in [1982] makes it a fascinating story. In this case, intrinsic and extrinsic supports are intertwined as in no other. Zermelo, in his passionate defense, cites both. He begins:

> ... how does Peano [one of Zermelo's critics] arrive at his own fundamental principles and how does he justify their inclusion...? Evidently by analyzing the modes of inference that in the course of history have come to be recognized as valid and by pointing out that the principles are intuitively evident [intrinsic] and necessary for science [extrinsic]—considerations that can all be urged equally well in favor of [the Axiom of Choice]. [1908, p. 187]

First the intrinsic supports predominate:

> That this axiom, even though it was never formulated in textbook style, has frequently been used, and successfully at that, in the most diverse fields of mathematics, especially in set theory, by Dedekind, Cantor, F. Bernstein, Schoenflies, J. König and others is an indisputable fact... Such an extensive use of a principle can be explained only by its *self-evidence*, which, of course, must not be confused with its provability. No matter if this self-evidence is to a certain degree subjective—it is surely a necessary source of mathematical principles ...

(Zermelo [1908, p. 187]. See also Fraenkel, Bar-Hillel and Levy [1973, p. 85].) Early set theorists did indeed use Choice implicitly, and the continuing difficulty of recognizing such uses is poignantly demonstrated by Jordain's persistent and ill-starred efforts to prove the axiom (see Moore [1982, §3.8]). Ironically, Choice was even used unconsciously by several French analysts who were officially its severest critics: Baire, Borel and Lebesgue (see Moore [1982, §§1.7 and 4.1]).[5]

[5]The referee indicates that the Paris school did eventually distinguish what they considered acceptable versions of Choice from the unacceptable ones, and that they were also the first to formulate the principle of Dependent Choice, an important tool in the presence of full Determinacy (see [BAII, §V]).

The debates over the intrinsic merits of the axiom centered on the opposition between existence and construction. Modern set theory thrives on a realistic approach according to which the choice set exists, regardless of whether it can be defined, constructed, or given by a rule. Thus:

> In many cases, it appears unlikely that one can *define* a choice function for a particular collection of sets. But this is entirely unrelated to the question of whether a choice function *exists*. Once this kind of confusion is avoided, the axiom of choice appears as one of the least problematic of the set theoretic axioms.

(Martin [SAC, pp. 1–2]) *Iterative conception* theorists seem also to lean on this *realism* rather than on the iterative picture itself (see Boolos [1971, pp. 501–502]; Shoenfield [1977, pp. 335–336]). One might also revert to *Cantorian finitism* (see Hallet [1984, p. 115]). (I will discuss another rule of thumb supporting Choice in II.2 below.)

Zermelo goes on to emphasize extrinsic supports:

> But the question that can be objectively decided, whether the principle is *necessary for science*, I should now like to submit to judgment by presenting a number of elementary and fundamental theorems and problems that, in my opinion, could not be dealt with at all without the principle of choice.

[1908, pp. 189–190]. He then describes seven theorems that depend on the Axiom of Choice, including the fact that a countable union of countable sets is countable, as well as two examples from analysis. Since then it has become clear that the Axiom of Choice and its equivalents are essential not only to set theory but to analysis, topology, abstract algebra and mathematical logic as well.

To take just one example, Moore [1982, §4.5] describes the axiom's growing importance in algebra during the 20s and 30s. In 1930, van der Waerden published his *Modern Algebra*, detailing the exciting new applications of the axiom. The book was every influential, providing Zorn and Teichmüller with a proving ground for their versions of choice, but van der Waerden's Dutch colleagues persuaded him to abandon the axiom in the second edition of 1937. He did so, but the resulting limited version of abstract algebra brought such a strong protest from his fellow algebraists that he was moved to reinstate the axiom and all its consequences in the third edition of 1950. Moore summarizes, "Algebraists insisted that the axiom had become indispensable to their discipline" (p. 235). And they were not alone.

Nowadays, intrinsic arguments for Choice in terms of intuitiveness or obviousness go hand-in-hand with extrinsic arguments in terms of its indispensability. Modern mathematics has sided firmly with Zermelo:

> ... no one has the right to prevent the representatives of productive science from continuing to use this "hypothesis"—as one may call it for all I care—and developing its consequences to the greatest extent ... We need merely separate the theorems that necessarily require the axiom from those that can be proved without it in order to delimit the whole of Peano's

> [choiceless] mathematics as a special branch, as an artifically mutilated science, so to speak ... principles must be judged from the point of view of science, and not science from the point of view of principles fixed once and for all. (p. 189)

I.8. *Replacement.* Early hints of the Axiom of Replacement can be found in Cantor's letter to Dedekind [1899] and in Mirimanoff [1917], but it does not appear on Zermelo's list in [1908]. This omission is due to his reductionism, that is, his belief that theorems purportedly about numbers (cardinal or ordinal) are really about sets. Since von Neumann's definition of ordinals and cardinals as sets, this position has become common doctrine, but Zermelo first proposed his axioms in the context of Cantor's belief that ordinal and cardinal numbers are separate entities produced by abstraction from sets. So, while Cantor sometimes stated the well-ordering theorem in the form "Every set is isomorphic to some ordinal number", Zermelo preferred the form "Every set can be well-ordered". As a result, he had no need for Replacement. (See Hallet [1984].)

Around 1922, both Fraenkel and Skolem noticed that Zermelo's axioms did not imply the existence of

$$\{N, \mathscr{P}(N), \mathscr{P}(\mathscr{P}(N)), \ldots\}$$

or the cardinal number $\aleph_\omega$. These were so much in the spirit of informal set theory that Skolem proposed an Axiom of Replacement to provide for them. It then took von Neumann to notice the importance of Replacement for the ordinal form of the well-ordering theorem, as well as in the justification of transfinite recursion.[6] Zermelo included it (in his second order version) in [1930].

Replacement is made to order for the *limitation of size* theorists:

> ... our guiding principle ... is to admit only axioms which assert the existence of sets which are not too "big" compared to sets already ascertained. If we are given a set *a* and a collection of sets which has no more members than *a* it seems to be within the scope of our guiding principle to admit that collection as a new set. We still did not say exactly what we mean by saying that the collection has "no more" members than the set *a*. It turns out that it is most convenient to assume that the collection has "no more" members than *a* when there is a "function" which correlates the members of *a* to all the sets of the collection ...

(Fraenkel, Bar-Hillel and Levy [1973, p. 50]). *The iterative conception* does less well because the only way to guarantee stages large enough to cover the range of the given function is to assume a version of Replacement in the theory of stages (see Shoenfield [1977, p. 326]); Boolos [1971, p. 500]).

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[6] Von Neumann actually used a stronger principle based on *limitation of size*, namely, "A collection is too large iff it can be put in one-to-one correspondence with the collection of all sets." This implies Separation, Replacement, Union and Choice (even Global Choice). Gödel found von Neumann's axiom attractive because it takes the form of a maximal principle (compare *maximize* in II.2 below): anything that can be a set, is. See Moore [1982, pp. 264–265].

On the extrinsic side stand the deep set theoretic theorems noted in the paragraph before last:

> ... the reason for adopting the axioms of replacement is quite simple: they have many desirable consequences and (apparently) no undesirable ones.

(Booles [1971, p. 500]) Still, the consequences noted are all within set theory; there is nothing like the broad range of applications found in the case of Choice. Recently, however, Martin used Replacement to show that all Borel sets are determined (see Martin [1975]). Earlier work of Friedman establishes that this use of Replacement is essential (see Friedman [1971]). Thus Replacement has consequences in analysis, consequences even for the simple sets of reals favored by the French analysts. Furthermore, these consequences are welcome ones, as we shall see in [BAII, §V].

Let me end this survey here, leaving the interested reader to the more informed works of the mathematical historians. I think enough has been said to demonstrate that from the very beginning, the process of adopting set-theoretic axioms has not been a simple matter of noting down the obvious. Rather, the axioms we now hold to be self-evident were first justified by reference to vague rules of thumb and purely extrinsic consequences, in addition to intrinsic evidence. The arguments offered for the new axioms are no different. But first we should pause to look at the problem that makes new axioms vital.

§II. The continuum problem. Cantor first stated his continuum hypothesis in 1878:

> The question arises ... into how many and what classes (if we say that [sets of reals] of the same or different [cardinality] are grouped in the same or different classes respectively) do [sets of reals] fall? By a process of induction, into the further description of which we will not enter here, we are led to the theorem that the number of classes is two.

(See Cantor [1895, p. 45].) The nature of this "process of induction" is never made clear, but Hallet reconstructs it from the contents of a letter Cantor wrote to Vivanti in 1886 (see Hallet [1984, pp. 85–86]). There Cantor comments on Tannery's purported proof of the continuum hypothesis:

> He believed he had given a proof for the theorem first stated by me 9 years ago that only two [equivalence] classes [by cardinality] appear among linear pointsets, or what amounts to the same thing, that the [cardinality] of the linear continuum is just the *second*. However, he is certainly in error. The facts which he cites in support of this theorem were all known to me at the time, as anyone can see, and form only a part of that induction of which I say that it led me to that theorem. I was convinced at that time that this induction is *incomplete* and I still have this conviction today.
>
> ... the theorem to be proved is
>
> $$\mathfrak{c} = \aleph_1.$$

> The facts on which Herr T. believes he can base the theorem are only these:
>
> $$n + \aleph_0 = \aleph_0, \aleph_0 + \aleph_0 = \aleph_0, \aleph_0 \cdot n = \aleph_0, \aleph_0^2 = \aleph_0,$$
> $$\aleph_0^n = \aleph_0; \aleph_0^{\aleph_0} = \mathfrak{c}, 2^{\aleph_0} = \mathfrak{c}, 3^{\aleph_0} = \mathfrak{c}, \ldots, n^{\aleph_0} = \mathfrak{c}.$$
>
> These facts suggest the conjecture that $\mathfrak{c}$ should be the power $\aleph_1$ following next after $\aleph_0$; but they are a long way from furnishing a proof for it.

Perhaps these facts seem even less persuasive today.

II.1. *Cantor's views.* Cantor's writings suggest two other reasons he might have had for believing the continuum hypothesis.[7] In 1874, Cantor proved the first version of his famous theorem: no countable sequence of elements from a real interval can exhaust that interval. In 1883, he proved that there are more countable ordinals than finite ordinals, and that any infinite set of countable ordinals is either countable or equinumerous with the set of all countable ordinals. Three things must have struck Cantor here: first, the two proofs of nondenumerability are similar (the usual diagonal argument for the nondenumerability of the reals came only in 1891), which produces an analogy between the reals and the countable ordinals (see below); second, the property proved for infinite subsets of the countable ordinals is exactly what CH conjectures for the reals; third, that the CH could now be formulated as "the reals and the countable ordinals are equinumerous."

Cantor apparently found evidence for the CH in the structural similarities revealed by the two proofs of nondenumerability. In particular, he came to see the reals and the countable ordinals as generated by similar processes from similar raw materials; in both cases, one begins with a countable set (the rationals and the finite ordinals, respectively) and one considers countably infinite rearrangements (Cauchy sequences and well-orderings, respectively). This analogy suggests that the two sets may also share the same cardinal number. Add to this the discovery that the set of countable ordinals has exactly the property Cantor expected to hold for the reals, and the CH in its new form seems a fairly natural conjecture (see Hallet [1984, pp. 74–81]).

Of course, wherever there are analogies there are both similarities and dissimilarities. What makes the 1874 proof of nondenumerability go through is the fact that any bounded sequence of reals approaches a limit; likewise, the 1883 proof of nondenumerability depends on the fact that any countable sequence of countable ordinals has a countable ordinal as its supremum. Still, as Hallet points out (p. 81), the topologies underlying these limit properties are not really all that similar.

A second reason Cantor may have had for believing the continuum hypothesis is based on the Cantor-Bendixon theorem of 1884, that is, the result that every closed infinite set of reals is countable or has a perfect subset, and hence, that CH is true for closed sets of reals. At the end of the paper in which this result is proved, he promises a proof of the same result for nonclosed sets of reals. He may have believed at one time that the proof itself could be generalized, and in fact, it can to a certain extent. I will take up the idea that these partial results constitute evidence for CH in

[7] Apparently, the term "continuum hypothesis" was first used by Bernstein in 1901. See Moore [1982, p. 56].

II.3.1. In any case, it is clear that for a while he hoped to establish CH by finding a closed set of cardinality $\aleph_1$. Such a set would be nondenumerable, so by the Cantor-Bendixon theorem, it would have the cardinality of the continuum. But that cardinality is $2^{\aleph_0}$, so CH is true. Working along these lines, in 1884 he wrote:

> I am now in possession of an extremely simple proof for the most important theorem of set theory, that the continuum has the [cardinality] of the [set of countable ordinals] ... you see that everything reduces to defining a closed set having [cardinality $\aleph_1$]. When I have sorted it out, I will send you the details.

(See Hallet [1984, p. 92]; Moore [1982, p. 43].) Of course, the details were never sorted out.[8]

For the record, during the prehistory of CH (that is, before the consistency and independence results), opinion seems to have been divided. Hilbert and Jourdain were both in favor (Moore [1982, pp. 55, 63]), though Hilbert apparently did not expect it to be provable in ZFC alone (Wang [1981, p. 656]). König attempted to prove it false, but only because he felt the reals could not be well-ordered at all (Moore [1982, §2.1]). Finally, Gödel cites Lusin and Sierpiński as tending to disbelieve it for reasons closer to his own ([1947/64, p. 479]).

II.2. *Consistency and Independence.* A wag once suggested that if only Gödel had announced having *proved* the continuum hypothesis, instead of its mere *consistency*, there would be no more continuum problem. Strangely enough, Gödel does almost exactly that in [1938]. Of the Axiom of Constructibility, from which he did prove CH, he writes:

> The proposition ... added as a new axiom, seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way. (p. 557)

By [1944], however, he has changed his mind and come around to the view now so strongly associated with his name:

> [The] axiom [of constructibility] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set ... (pp. 478–479)

Perhaps Godel's new opinion of $V = L$ was also influenced by his developing belief in the falsity of CH (see II.3.3 below).

We see here the statement of a new rule of thumb, namely *maximize.*[9] This rule is often associated with the *iterative conception* in two more specific forms:

> Intrinsic necessity depends on the concept of iterative model. In a general way, hypotheses which purport to enrich the content of power sets ... or to introduce more ordinals conform to the intuitive model. We believe that the collection of all ordinals is very 'long' and each power set (of an infinite set)

[8] Cantor's final attempt at proving the continuum hypothesis involved a new method of decomposing arbitrary sets of reals. See Moore [1982, pp. 43–44], and Hallet [1984, p. 103].

[9] Recall the earlier hint of this rule in Godel's reaction to von Neumann's axiom.

> is very 'thick.' Hence, any axioms to such effects are in accordance with our intuitive concept.

(Wang [1974, p. 553]) For example, the Axiom of Choice is widely thought to contribute to the "thickness" of the power set (see e.g. Gödel in Moore [1982, p. 265]; Drake [1974, p. 12]). I will take up the question of postulating more ordinals in the next section, but for now it is clear that the restriction to definable subsets at each stage can be seen as an unwelcome curtailment of the full power set. The view that $V = L$ contradicts *maximize* is widespread (see e.g. Drake [1974, p. 131]; Moschovakis [1980, p. 610]; Scott [1977, p. xii]).[10]

There are also extrinsic reasons for rejecting $V = L$, most prominently that it implies the existence of a Δ^1_2 well-ordering of the reals, and hence that there is a Δ^1_2 set which is not Lebesgue measurable.[11] It can be proved in ZFC that every Borel well-ordering of the reals is countable. A Δ^1_2 set can be obtained from a Borel set by one application of projection followed by one application of complement. Many find it implausible that a set as complex as a well-ordering of the real numbers could be generated by such simple operations.[12] The Axiom of Choice guarantees that there is such a well-ordering, but the proofs are highly nonconstructive, so it is considered implausible that the well-ordering should be definable at all (see Moschovakis [1980, p. 276]; Wang [1974, p. 547]; Martin [1976, p. 88] and [PSCN, p. 2]). Further extrinsic evidence against $V = L$ will be discussed in II.3.1, below, and in [BAII, §V].

After his proof of the consistency of CH, Godel conjectured that it is independent as well. The axioms of ZFC, he argues, are true in V and in L, so one can hardly expect to decide the numerical question of the size of the continuum until one has settled "what objects are to be numered, and on the basis of which one-to-one correspondences." Even if one believes that $V = L$,

> ... he can hardly expect more than a small fraction of the problems of set theory to be solvable without making use of this, in his opinion essential, characteristic of sets. [1947/64, p. 478]

Finally, in 1963, Cohen proved him right (see his [1966]).[13]

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[10] I should not suggest that no one supports the adoption of $V = L$ as an axiom; sentiment in favor can be found (see e.g. Fraenkel, Bar-Hillel and Levy [1973, pp. 108–109]; Devlin [1977, p. iv]). Reasons usually given are that it is simple and safe (see Moschovakis [1980, p. 609]), and that it provides answers to a great many outstanding problems. Discussion below and in [BAII, §VI] will suggest that these answers are "in the wrong direction", but that opinion is surely open to debate. Despite all this, I will stick to the anti-($V = L$) line because it is favored by the Cabal group.

[11] *Notation*: Σ^0_1 is the class of open sets of reals; Π^0_α is the class of all complements of Σ^0_α sets of reals; $\Sigma^0_{\alpha+1}$ is the class of all countable unions of Π^0_α sets; and $\Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha$. All these together are the Borel sets. Further, Σ^1_1 is the class of all projections of closed sets; Π^1_α is the class of all complements of Σ^1_α sets; $\Sigma^1_{\alpha+1}$ is the class of projections of Π^1_α sets; and Δ^1_α is $\Sigma^1_\alpha \cap \Pi^1_\alpha$. These are the projective sets. In 1917, Souslin proved that the Borel sets are the Δ^1_1 sets. Finally, if R is a well-ordering of the reals, then Fubini's theorem implies the R is not Lebesgue measurable.

[12] This way of putting the implausibility was suggested by Matt Foreman.

[13] See Scott [1977] and Wang [1981] for some discussion of why the independence proof was so long in coming.

While Gödel's result had a temporary discouraging effect on research in set theory (for fear that the problem in question was in fact undecidable), Cohen's invention of the forcing method led to a boom (see Martin [1976, pp. 82–83]). While the truth of CH in the constructible universe has had some influence on opinion as to its truth or falsity (see II.3.4 below), the relevance of forcing models to that question is much less clear. The plethora of different models moved Cohen himself to a version of formalism (see his [1966] and [1971]), but Scott, another innovator in the early development of forcing, writes:

> I myself cannot agree, however. I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms; but the models are all just models of the first-order axioms, and first-order logic is weak. I still feel that it ought to be possible to have strong axioms which would generate these types of models as submodels of the universe, but where the universe can be thought of as something absolute. [1977, p. xiv]

(See also Kanamori and Magidor [1978, p. 109].) Perhaps the association of CH with the restrictive $V = L$, combined with the development of this striking new technique for adding extra real numbers to models, led some to agree with Gödel that CH is false in the absolute real world.

II.3. *Informed opinion.* Despite the results of Gödel and Cohen, there remain set theorists who feel the CH is a real question, the sort of thing that is either true or false in the real world of sets. Various arguments for and against have been bandied about in their ranks. The purpose of this subsection is to summarize the most prominent of these.

II.3.1. *Partial results (in favor).* Recall that Cantor may have expected the proof of the Cantor-Bendixon theorem (that CH holds for closed sets of reals) to generalize to all sets of reals. This program was carried forward by Young in 1906 to a subset of the Π^0_2 sets, then by Hausdorff in 1914 to all Π^0_2 sets, and finally, by Hausdorff again in 1916 to all Borel sets. Still, Hausdorff himself was reluctant to count these results as evidence for the CH:

> If we knew for all sets what we know for closed and Π^0_2 ... then ... the continuum hypothesis would be decided. However, in order to see how far we still are from this goal, it is sufficient to recall that the system of sets closed or Π^0_2 forms only a vanishingly small part of the system of all point sets.

Even after the proof had been extended to all Borel sets, he continued:

> Thus the question of power is clarified for a very inclusive category of sets. Nevertheless, one can scarcely see this as a genuine step towards the solution of the continuum problem, since the Borel sets are still very specialized, and form only a vanishingly small subsystem.

(Both translations are due to Hallet [1984, p. 107].) Of course there are $2^{2^{\aleph_0}}$ set of reals, only $2^{\aleph_0}$ of which are Borel.

Even more damaging to the interpretation of these results as evidence in favor of the CH is something Hausdorff apparently did not realize at the time, namely that his proofs could be strengthened to the form of the original Cantor-Bendixon result, that is, that every infinite Borel set is either countable or contains a perfect subset. In 1916, Alexandroff proved the theorem in this form, and in 1917, Souslin extended it to Σ^1_1 sets. (In the presence of a measurable cardinal, this pattern can be extended to Σ^1_2. See §IV below.) The trouble arises from Bernstein's proof that there are uncountable sets of reals without perfect subsets. Thus these proofs that CH holds for restricted classes of sets all depend on establishing a stronger property, the perfect subset property, that cannot hold for all sets of reals (see Martin [1976, p. 88]; Hallet [1984, pp. 103–110]). For this reason, the technique cannot be fully generalized.

In Cantor's defense, it should be noted that he was probably unaware of the existence of uncountable sets without perfect subsets. Most of the sets of reals Cantor worked with were Σ^1_1 at worst. Furthermore, Bernstein's proof, published in 1908, made essential use of the Axiom of Choice. Though Cantor often used that axiom, he did so to form orderings, or to make simultaneous choices from many order types or cardinalities, not to form sets of reals, so he may well not have noticed this possibility.

In 1925, Lusin wondered whether every infinite Π^1_1 set is either countable or contains a perfect subset. He writes:

> My efforts towards settling this question have led to an unwelcome result:
> we do not know *and will never know*...

(Translation due to Hallet [1984, p. 108]). This may sound overly dramatic, but in a sense, Lusin was right, for the Axiom of Constructibility implies the existence of an uncountable Π^1_1 set with no perfect subset, while other hypotheses imply the opposite (see [BAII, section §V]). That such a "pathology" should occur so low in the projective hierarchy is considered another extrinsic disconfirmation of $V = L$ (e.g. Wang [1974, p. 547]).

II.3.2. *The effectiveness of* CH (*in favor*). The generalized CH is an extremely simple and powerful assumption that immediately settles all questions of cardinal arithmetic. Furthermore, it allows any power set to be well-ordered in such a way that every initial segment is no bigger than the original set. This facilitates many complex constructions, such as saturated models of every regular cardinality. Sierpiński's book *Hypothèse du Continu* deduces 82 propositions from the CH. In stark contrast, Martin and Solovay remark [1970] that not a single one of these 82 propositions is known to be decided by the negation of the continuum hypothesis.

II.3.3. *Gödel's counterintuitive consequences* (*against*). In [1947/64], Gödel argues that CH is false because it has certain "highly implausible consequences" (p. 479). Several of these assert the existence of sets of reals of cardinality $2^{\aleph_0}$ with strong "smallness" properties. For example, a subset of the unit interval is called "absolute zero" if it can be covered by any countable collection of intervals. If covering is only required when the intervals are of equal length, then the set would have Lebesgue measure zero, but would not necessarily be absolute zero. Thus Cantor's discontinuum has Lebesgue measure zero, but is not absolute zero, because it cannot be covered by countably many intervals of length $1/3^n$. In fact, no perfect set can be absolute zero, and Borel conjectured that no set of size $2^{\aleph_0}$ could be. The CH implies that there is such a set.

Commentators were quick to point out that many consequences of the set-theoretic reduction of the continuum that do not depend on CH are similarly counterintuitive, for example, Peano's space-filling curve. Gödel insists that his examples are not of this sort, because in those cases:

> ... the appearance [of counterintuitiveness] can be explained by a lack of agreement between our intuitive geometrical concepts and the set-theoretical ones occurring in the theorem. (p. 480)

While we all might be surprised at Peano curves or the uncountable Cantor set of measure zero, this surprise is presumably based on exactly the clash Gödel mentions: a disagreement between our geometric intuition and our set-theoretic geometry, or a vague feeling that sets large in cardinality should not also be small in measure. But Gödel is basing his reactions on something else.

What then? Wang suggests that

> ... it cannot be excluded that someone might have such intimate knowledge so that, for example, he can separate out errors coming from using the pre-set-theoretical intuition.

[1974, p. 549]. He reminds us of Gödel's view that all intuition must be cultivated. It seems to me more likely that Gödel had in mind some form of peculiarly set-theoretic intuition not connected with pre-set-theoretic geometry. In either case, we are left with Gödel's bare claims, because even our best set theorists do not share these "intuitions":

> While Gödel's intuitions should never be taken lightly, it is very hard to see that the situation *is* different from that of Peano curves, and it is even hard for some of us to see why the examples Gödel cites are implausible at all.

(Martin [1976, p. 87]; see also Martin and Solovay [1970, p. 177]).

Gödel apparently did make at least one attempt to axiomatize his views on the continuum. It appears in Ellentuck [1975] and takes its cue from a conjecture of Borel. Suppose that the functions from ω to ω are ordered so that $\mathbf{f}$ is less than $\mathbf{g}$ if and only if $\mathbf{f}(n)$ is always less than $\mathbf{g}(n)$ after a proper initial segment. Borel conjectured that there is a set S of size $\aleph_1$ which is cofinal in this ordering. The "square axiom", A, is just this conjecture; the "rectangle axioms", A_n, are generalizations of the square axiom to functions from ω_n to ω. Gödel agreed with Borel on the plausibility of A; his hope was that the A_n's could be justified by analogy with A, and that they would set bounds on the size of the continuum.

Now $\aleph_2$ is the only value for $2^{\aleph_0}$ that is known to be consistent with the nonexistence of absolute zero sets. Furthermore, A_2 implies that $2^{\aleph_0} \leq \aleph_2$. Thus it seems Gödel must have suspected that $2^{\aleph_0} = \aleph_2$. Unfortunately, the plan to derive this from a theory involving the rectangle axioms was ruined with the discovery that A_1 implies CH. Alternate versions of the square axiom turned out to be relatively consistent with a wide range of values of $2^{\aleph_0}$, and A itself implies the existence of an absolute zero set of size $\aleph_1$. While this is perhaps not so counterintuitive as an absolute zero set of size $2^{\aleph_0}$, it must have been an unwelcome result. Thus this effort of Gödel's to formalize his intuitions about the continuum was quite unsuccessful.[14]

II.3.4. CH *is restrictive* (*against*). As mentioned earlier, perhaps some of the reason CH is felt to be restrictive is because it is true in L. If this line of thought is to have any force, it must first meet a difficult challenge, namely that the Axiom of Choice, generally regarded as a maximizing principle in itself, is also true in L. If L is so impoverished, why does the additional assumption of Choice have no maximizing effect? (It doesn't, because it is not an additional assumption, after all.)

I think the answer to this question is not so difficult. Choice is true in L because there is a definable well-ordering of the constructible universe. This reverses the intuitive order of things. Why is a given set well-orderable? Because an element can be chosen for each ordinal until the set is exhausted. Why should such choices be possible? Because *realism* and *maximize* guarantee the existence of a choice set. Thus the well-ordering principle derives from Choice and not vice versa.

The maximizing force of Choice lies in its implying the existence of complex, probably undefinable sets like a well-ordering of the reals, a non-Lebesgue-measurable set, and an uncountable set with no perfect subset. That the Axiom of Constructibility forces such sets far down into the simple projective sets counts as extrinsic evidence against it. Thus the Axiom of Choice is true in L, but it does not do any maximizing work because it is true for the wrong reason. It is not true because there are complex sets; rather it is true because there is an artificially simple well-ordering.

Now what about CH? Is it truly a maximizing principle that just happens to be true in the restricted world of L because its maximizing force is masked by some unrelated pathology? For what it's worth, I see nothing in the proof of CH in L that suggests this. CH is true in L because all the constructible subsets of ω appear in L_{ω_1}, and L_{ω_1} has small cardinality. But why is that cardinality small? Because the limited procedure of subset formation in L only allows at most one new element for every formula and finite sequence of parameters. Thus CH is true in L because the formation of subsets is artificially restricted, not because some other pathological condition in L is robbing it of its maximizing force.[15]

[14]The history of the square and rectangle axioms is described in more detail in Moore [1982].

[15]I should note that J. Friedman [1971] argues that GCH is a maximizing rather than a restrictive principle. He does so by showing it equivalent to what he calls the "generalized maximizing principle," namely, the assumption that every "local universe" contains all its smaller-cardinality subsets. (Note the similarity to von Neumann's maximizing principle above.) The problem is that a "local universe" is defined as a collection closed under Pairing, Union and Replacement. Obviously Replacement is being maximized at the expense of Power Set. Thus Friedman is right that:

> A fundamental question is whether the generalized maximizing principle maximizes these operations [Pairing, Union and Replacement] at the expense of the power set operation.

But perhaps he is less than candid when he claims:

> The Generalized Maximizing Principle says nothing explicit about the power set operation, but as an afterthought, GCH follows from it. (p. 41)

Another way of stating the idea that CH is restrictive is to insist that the continuum is somehow too complicated to be numbered by the countable ordinals. Drake presents a version of this view:

> Of course, many mathematicians do not feel that the cumulative type structure is a well-defined, unique object, and from this point of view the independence results may have to be considered the final word on the GCH. But there are also many mathematicians who feel that the cumulative type structure is real enough, in a sense, for the GCH, or at least the CH, to be a real question. It is worth noting that amongst these mathematicians, many feel that the GCH is just too *simple* to be right. Perhaps the following illustrates this feeling:... [1974, pp. 65–66]

He goes on to point out that $\aleph_1$ is the cardinal number of the collection of all countable well-ordering types, while $2^{\aleph_0}$ is the cardinal number of the collection of all countable linear ordering types.

> To say of a linear ordering that it is a well-ordering is a very strong requirement, so that there should be many more linear orderings than well-orderings ... (p. 66)

Of course, if CH were true, it would not be the first time that a difference in complexity was not mirrored in a difference in cardinality. It is hard to see why the CH should be interpreted as saying that there are not very many subsets of ω when it could just as easily be taken to say that there are lots of countable ordinals.

The question of how complexity matches up with cardinality is further muddled by results involving Martin's Axiom (see Martin and Solovay [1970]). Recall that many of the consequences of CH are made possible by the well-ordering of $2^{\aleph_0}$ with countable initial segments. Though Martin's Axiom is relatively consistent with $\aleph_1 < 2^{\aleph_0}$, it still guarantees that the cardinals smaller than $2^{\aleph_0}$ are well enough behaved to allow complicated constructions to go through. As a result, 79 of Sierpiński's 82 consequences of CH also follow from MA + ($\aleph_1 < 2^{\aleph_0}$) (with the natural modification that the countable/uncountable distinction is replaced by the less-than-$2^{\aleph_0}/2^{\aleph_0}$ distinction). Thus advocates of the view that the continuum is complex might wonder if large cardinality alone is enough to guarantee that complexity.

II.3.5 *Power Set is stronger than Replacement* (*against*). This position is Cohen's. As a formalist, Cohen realizes he should reject the question of the truth or falsity of CH [1966, p. 150], but he feels he cannot reject the same question concerning large cardinals:

> I, for one, cannot simply dismiss these question of set theory for the simple reason of their reflections in number theory. I am aware that there would be few operational distinctions between my view and the Realist position. [1971, p. 15]

Thus he is willing to speculate on the truth value of the CH from the realist point of view:

> A point of view which the author feels may eventually come to be accepted is that CH is *obviously* false. The main reason one accepts the Axiom of

> Infinity is probably that we feel it absurd to think that the process of adding only one set at a time can exhaust the entire universe... Now $\aleph_1$ is the set of countable ordinals and this is merely a special and the simplest way of generating a higher cardinal. The set C is, in contrast, generated by a totally new and more powerful principle, namely the Power Set Axiom. It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C. Thus C is greater than $\aleph_n$, $\aleph_\omega$, $\aleph_\alpha$ where $\alpha = \aleph_\omega$, etc. This point of view regards C as an incredibly rich set given to us by a bold new axiom, which can never be approached by any piecemeal process of construction. Perhaps later generations will see the problem more clearly and express themselves more eloquently. [1966, p. 151]

In this connection, recall that the *limitation of size* theorists had difficulties with Power Set but smooth sailing with Replacement. It should be noted that most set theorists who disbelieve CH think $2^{\aleph_0}$ is more likely to be very large, as Cohen indicates, than to be $\aleph_2$, as Gödel suggests.

II.3.6. *Finitism* (*against* GCH). Here the argument depends on analogy with the finite numbers, where $n + 1 = 2^n$ is true only for 0 and 1. This is felt by some to constitute an argument against the GCH, if not against the particular case of the CH (see Drake [1974, p. 66]).

II.3.7. *Whimsical identity* (*against* GCH). This argument depends on the same facts as the finitism argument, but it uses them in a different way. Notice that if the GCH were true, then $\aleph_0$ could be defined as that cardinal before which GCH is false and after which it is true (excepting 0 and 1, of course). But this identity would seem "accidental", like the identity between "human" and "featherless biped". While the physical universe might be too impoverished to falsify such accidental identities, the set-theoretic universe should be rich enough to rule them out. Therefore, GCH is false. (Kanamori and Magidor [1978, p. 104] and Martin [1976, p. 85] use whimsical identity arguments to support large cardinal axioms. See §III below.) Of course this line of argument faces considerable difficulties in explaining what is meant by "accidental", and how this particular identity can be seen to have that property.

II.3.8. *The delicate balance* (*against*). This argument is stated by Wang:

> Some set theorist states that if $\aleph_1 = 2^{\aleph_0}$, then there must be a surprisingly delicate balance between the reals and the countable ordinals. [1974, pp. 549–550]

As Wang goes on to point out, the balance must be delicate whatever the cardinality of the reals turns out to be. Indeed, it might seem more delicate if $2^{\aleph_0}$ were $\aleph_{17}$.

II.3.9. *Gupta's wager* (*against*). Gupta suggests, somewhat facetiously, that since $\aleph_1$ is only one among the proper-class-many values $2^{\aleph_0}$ might consistently take, it makes more sense to plump for not-CH.

II.3.10. *Freiling's darts* (*against*). Freiling [1986] suggests a thought experiment in which random darts are thrown at the real line. Suppose that a countable set $f(x)$ is associated with each real x. Now I throw two darts; the first hits a point x_0 and the second a point x_1. Given that a countable set is very sparsely distributed, the probability that my second dart will hit a member of $f(x_0)$ is vanishingly small. Thus, in all likelihood, the point x_1 is not a member of $f(x_0)$.

But the situation is symmetric: there is just as little reason to suppose that the first dart has hit a point in the set $f(x_1)$. Thus Freiling proposes that for every assignment of a countable set to each real, there are two reals neither of which is a member of the set assigned to the other. This rather innocuous-sounding statement turns out to be equivalent to not-CH.

A common objection to this line of thought is that various natural generalizations contradict the axiom of choice as well. For example, if Freiling's principle is modified to cover assignments of sets of any cardinality less than that of the reals, the result immediately implies that there is no well-ordering of the reals. Similarly if we are allowed to throw $\omega + 1$ darts. Members of the Cabal suggest that Freiling's hypotheses yields a picture more like that of the full AD-world (see [BAII, §V]) than of the choiceful universe V. Meanwhile, Freiling disputes the "naturalness" of these generalizations. He also points out that *one step back from disaster* could provide a rationale for accepting his axiom and rejecting its generalizations even if they are "natural."

II.3.11. *Not*-CH *is restrictive* (*in favor*). This argument uses the same general considerations as II.3.4 in support of the opposite conclusion. While established opinion among more mature members of the Cabal is against CH, younger members are sympathetic to this more recent argument and to the considerations raised in the next subsection. It has been suggested that the cut-off age is 40.

To see how not-CH can be considered restrictive, we imagine ourselves constructing the iterative hierarchy. By stage $\omega + 2$, we have the set of reals and we have a well-ordering of type $\aleph_1$. The question is whether or not a one-to-one correspondence between them is included at the next stage. Since it is consistent to do so, it would artifically restrict the power set operation to leave it out. The thinking behind II.3.4 sees CH as restricting the size of the power set of ω. From the present point of view, not-CH is a restriction of the power set operation at the next stage.[16]

II.3.12. *Modern forcing* (*in favor*). Practitioners of modern versions of forcing point out that it is much easier to force CH than not-CH; that is, that a wide variety of forcing conditions collapse $2^{\aleph_0}$. Since the addition of generic sets tends to make CH true, it is most likely true in the full richness of V itself.

I think this list includes most of the arguments standardly offered for and against the CH. It should be emphasized that few set theorists consider any of them conclusive, and even those with fairly strong opinions adopt a decidedly wait-and-see attitude toward CH. Let me turn now to the search for new axioms to settle the question.

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[16]Chris Freiling points out that an argument of similar form can be presented against the Axiom of Choice. Notice that a choice function for a countable partition of the reals can be coded as a single real. At stage $\omega + 1$, we have all the reals, so we also have all codes of choice functions for countable partitions. Any countable partition can be coded as a set of reals at stage $\omega + 2$. Thus the question, at stage $\omega + 2$, is whether or not to include a countable partition without a choice function.

§III. Small large cardinals—up from below. There are those who hold that the universe of sets is not sufficiently well-defined for the continuum question to have an unambiguous answer. Little can be done to rebut this position short of coming up with an unambiguous solution, so perhaps this question should be set aside pending further developments. A less reasonable view is that the consistency and independence proofs by themselves show that the CH poses a meaningless question. It is hard to see any justification for the implicit claim that the axioms of ZFC must be taken as the final word:

> Although the ZFC axioms are insufficient to settle CH, there is nothing sacred about these axioms...
>
> ... undecidability [of CH] from the axioms being assumed today can only mean that these axioms do not contain a complete description of [set-theoretic] reality.

(Martin [1976, p. 84]; Gödel [1947/64, p. 476])

Where are we to look for the new axioms that will make our description more complete? In [1946], Gödel suggests:

> ...stronger and stronger axioms of infinity...It is not impossible...that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of sets.

(p. 85)[17] Skolem's and Fraenkel's introduction of the Axiom of Replacement can be seen in this light as they specifically wanted to generate $\aleph_\omega$. Thus the suspicion that adding larger ordinals can produce new results about sets of reals is confirmed by Martin's proof of Borel determinacy (see Hallet [1984, p. 102]; Kreisel [1980, p. 60]).

The first such new axiom of infinity is the Axiom of Inaccessibles, asserting the existence of regular, strong limit cardinals. The existence of such cardinals was first suggested by Zermelo [1930]; the axiom itself was formulated by Tarski in [1938]. Gödel presents an intrinsic defense:

> These axioms show clearly, not only that the axiomatic system of set theory as used today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which only unfold the content of the concept of set explained [by the iterative conception]. [1947/64, pp. 476–477]

(See also Wang [1974, p. 554].) Of course, *maximize* presents a simple and immediate defense for the Axiom of Inaccessibles. Recall that this rule of thumb is actually a pair of admonitions: thicken the power set, and lengthen the class of ordinals. Axioms of infinity in general, and the Axiom of Inaccessibles in particular, clearly do the second of these.

The most commonly given argument is more closely tailored to the actual content of Inaccessibles (see e.g. Gödel [1947/64, p. 476]; Wang [1974, p. 554]; Drake [1974,

[17] This conjecture may seem less likely in light of Levy and Solovay's strong theorem [1967] on the stability of large cardinals under most forcing extensions. See §IV below.

pp. 267–268]). It depends on the widespread view that the universe of sets is too complex to be exhausted by any handful of operations, in particular by power set and replacement, the two given by the axioms of Zermelo and Fraenkel. Thus there must be an ordinal number after all the ordinals generated by replacement and power set. This is an inaccessible.[18] Similarly, the universe above a given point should not be exhausted by these two operations, so there is another inaccessible, and so on. Versions of *inexhaustibility* can also be used to defend the various hyperinaccessibles and Mahlo cardinals. All of these are generated by thinking of processes that build up larger ordinals from below.

The Axiom of Inaccessibles can also be defended by two other rules of thumb each incomparably stronger than *inexhaustibility*. The first of these is *uniformity*.[19] To understand the thrust of this rule, suppose that a certain interesting situation occurs at a low level of the iterative hierarchy. If no similar situation occurred in the remainder of the hierarchy, it would be as if the universe had lost its complexity at the higher levels, as if it had flattened out, become homogeneous and boring. *Uniformity* says that this does not happen, that situations similar to our chosen interesting one will recur at higher levels:

> We mean by uniformity a process of reasonable induction from familiar situations to higher orders, with the concomitant confidence in the recurring richness of the cumulative hierarchy.
>
> Uniformity of the universe of sets (analogous to the uniformity of nature): the universe of sets doesn't change its character substantially as one goes over from smaller to larger sets or cardinals, i.e., the same or analogous states of affairs reappear again and again (perhaps in more complicated versions).

(Kanamori and Magidor [1978, p. 104]; Wang [1974, p. 541]; see also Solovay, Reinhardt and Kanamori [1978], and Reinhardt [1974, p. 189]. Wang and Reinhardt attribute support for this principle to Gödel.) Thus, $\aleph_0$ is inaccessible, so there must be uncountable inaccessible cardinals. Otherwise, the universe would be sparse above $\aleph_0$, or change its character in an objectionable way.

Uniformity arguments often go hand-in-hand with *whimsical identity* arguments. In this case, for example, if there are no uncountable inaccessibles, then $\aleph_0$ can be defined as the inaccessible. But:

> It would seem rather accidental if $\aleph_0$ can be characterized by [this] property.

(Kanamori and Magidor [1978, p. 104]; see also Martin [1976, p. 85]). So there must be uncountable inaccessibles.

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[18] Of course, Replacement must be taken in Zermelo's second-order form.

[19] Solovay, Reinhardt and Kanamori [1978] and Kanamori and Magidor [1978] call this principle *generalization*, while Wang [1974] calls it *uniformity*. I will want to retain the first of these for a slightly different rule of thumb (see [BAII, §VI]). Hallet [1984, pp. 114–115] also connects *uniformity* to the views behind *cantorian finitism*, but not in the way suggested here.

Uniformity itself is sometimes defended on the basis of *Cantorian finitism*: the sequence of natural numbers continues to produce interesting complexities arbitrarily far out, so the sequence of transfinite ordinals should do the same. Unfortunately, the premise concerning the natural numbers is debatable. While the sequence of natural numbers does continue to produce, for example, arbitrarily large prime numbers, it may or may not continue to produce twin primes, and it definitely runs out of adjacent primes after 2 and 3, and even primes after 2. This highlights the delicacy of formulating the property to be projected. Projecting properties of $\aleph_0$ is similarly chancy. I will return to this point below, in connection with weakly compact cardinals.

The second powerful rule of thumb sometimes cited in support of Inaccessibles is *reflection*: the universe of sets is so complex that it cannot be completely described; therefore, anything true of the entire universe must already be true of some initial segment of the universe. In other words, any attempt to uniquely describe V also applies to smaller R_α's that "reflect" the property ascribed to V.[20] In particular, V is closed under the operations of replacement and power set, so there is an R_κ which is also so closed. Then κ is an inaccessible. Similarly, V is closed under replacement and power set above this κ, so there is another inaccessible, and so on.

Hallet [1984, pp. 116–118], traces *reflection* to Cantor's theory that the sequence of all transfinite numbers is absolutely infinite, like God. As such, it is incomprehensible to the finite human mind, not subject to mathematical manipulation. Thus nothing we can say about it, no theory or description, could single it out; in other words, anything true of V is already true of some R_α. Reinhardt [1974, p. 191] expresses a similar sentiment, though without the reference to God. A related view is that the universe of set theory is "ever-growing", so that our attempt to speak of "all sets" actually refers to "temporary" partial universes that "approximate" the universe of all sets (Fraenkel, Bar-Hillel and Levy [1973, p. 118]; see also Parsons [1974] and Wang [1974, p. 540]). Discussions of this sort characteristically emphasize the indefiniteness or incomprehensibility or ineffability of V.

Martin strikes a somewhat different note:

> Reflection principles are based on the idea that the class ON of all ordinal numbers is so large that, for any reasonable property P of the universe V, ON is not the first stage α such that R_α has P. [1976, pp. 85–86]

Here the emphasis is on the largeness and complexity of the class of ordinals rather than some mysterious indefinability V; it is not that V is so inscrutable that nothing can describe it, but that ON is so vast that whatever happens at the top must already have happened before.

In any case, *reflection* is probably the most universally accepted rule of thumb in higher set theory (in addition to references already cited, see Solovay, Reinhardt and Kanamori [1978, p. 104], and many others). It is partially confirmed by weak, single formula versions that are provable in ZFC (see Levy [1960]). More powerful applications attempt to use stronger properties involving infinite sets of formulas,

[20] Notice that *reflection* implies *inexhaustibility*.

and/or higher order properties, while avoiding "nonstructural" properties, like "$x = V$", which lead to contradiction.

It should be mentioned that the Axiom of Inaccessibles also has a few extrinsic merits. It implies that ZFC has a standard model in the iterative hierarchy, and thus, that ZFC is consistent. This last is an arithmetic fact, and the Axiom of Inaccessibles, like other axioms of infinity, also implies the solvability of new Diophantine equations. (These facts are often cited. See e.g. Gödel [1947/64, p. 477], the quotation from Cohen [1971, p. 15], cited in II.3.5 above, and Kanamori and Magidor [1978, p. 103], to name a few.) In addition, there are the impressive relative consistency results of Solovay [1970]. Assuming a model of "ZFC + The Axiom of Inaccessibles", Solovay uses forcing to collapse the inaccessible and obtains models of ZFC in which all or many sets of reals are Lebesgue measureable ("many" means those constructible from the reals). Thus, these conditions are refutable only in the unlikely event that inaccessibles are refutable. Solovay's theorem:

> ... even today rivals any other as the most mathematically significant result obtained by forcing since Cohen's initial work.

(Kanamori and Magidor [1978, pp. 204–205]). And it presupposes the consistency, if not the existence, of an uncountable inaccessible.

There are larger small large cardinals, but nothing new appears in the usual defenses.[21] An exception is weakly compact cardinals, from discussions of which two morals can be derived. These cardinals can be defined in terms of a generalization of Ramsey's theroem; that is, κ is weakly compact iff every partition of the two-element subsets of κ into two groups has a homogeneous set of size κ. Because of Ramsey's theorem on $\aleph_0$, the existence of an uncountable weakly compact cardinals can be defended by *uniformity* or by *whimsical identity*. The first point of interest is that the proof of Ramsey's theorem also gives a homogeneous set for partitions of n-element set into m groups, but this property cannot be consistently generalized to an uncountable cardinal (see Drake [1974, p. 315]). This dramatically spotlights the difficulty of knowing when *uniformity* and *whimsical identity* can be applied without ill effect.

Second, the property of weakly compactness is equivalent to the compactness of the language $L_{\kappa\kappa}$, and to a certain tree property, and to an indescribability property, and to several other natural properties (see Drake [1974, §10.2]). This convergence has led some writers to *diversity*, another rule of thumb:

> It turned out that weak compactness has many diverse characterizations, which is good evidence for the naturalness and efficacy of the concept.

(Kanamori and Magidor [1978, p. 113]). Recall that similar arguments were once given for the naturalness of the notion of general recursiveness.

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[21]Though extrinsic defenses are nothing new, Harvey Friedman has extended the range of such defenses for small large cardinals. His [1981] contains nonmetamathematical statements, statements not involving such "abstract" notions as uncountable ordinals or arbitrary sets of reals, which are provable with and not without the assumption of Mahlo cardinals. See Drake [1974] for an account of Mahlo cardinals, and Stanley [1985] for a description of recent extensions of Friedman [1981].

§IV. Measurable cardinals. Measurable cardinals were introduced by Ulam in [1930], where he proved that they are inaccessible. They are now known to be much larger than that, larger than all the hyperinaccessibles, Mahlos and weakly compacts. Indeed, because of their power, they are probably the best known large cardinals of all. The voice of caution reminds us that they were invented by the same fellow who invented the hydrogen bomb.[22]

Unlike the small large cardinals suggested by *inexhaustibility*, measurable cardinals are not usually held to follow naturally from the concept of set or the nature of the iterative hierarchy:

> ... that these axioms are implied by the general concept of set in the same sense as Mahlo's has not been made clear yet.

(Gödel [1947/64, pp. 476–477])[23] Some wish for an *inexhaustibility* defense:

> What we would really like to do (but are presently unable to do) is to reformulate the definition of a measurable cardinal to look like this: κ is measurable iff R_κ is closed under certain operations.

(Shoenfield [1977, p. 343]) Others are more harsh:

> Also there are axioms such as that of the Measurable Cardinal which are more powerful than the most general Axiom of Infinity yet considered, but for which there seems absolutely no intuitively convincing evidence for either rejection or acceptance.

(Cohen [1971, pp. 11–12]) Against this we should point out that the very fact that the Axiom of Measurable Cardinals implies the existence of so many small large cardinals provides evidence based on *maximize.*

The rule of thumb most commonly cited in discussions of measurable cardinals is *uniformity* (see Wang [1974, p. 555]; Drake [1974, p. 177]; Kanamori and Magidor [1978, pp. 108–109]; Martin [PSCN, p. 8]). A measure on a cardinal κ is a division of its subsets into large and small in such a way that κ is large, $\varnothing$ and singletons are small, complements of large sets are small and small sets large, and intersections of fewer than κ large sets remain large. A measure on $\aleph_0$ is formed by extending the cofinite filter to an ultrafilter. Thus $\aleph_0$ is measurable, so *uniformity* implies that there are uncountable measurable cardinals. To apply *whimsical identity* instead, notice that if there were no uncountable measurable cardinals, then $\aleph_0$ could be defined as the infinite measurable cardinal. (2 is also measurable.)

Unfortunately, as pointed out in connection with weakly compact cardinals, *uniformity* can lead to inconsistencies. Thus in cases where this is the main rule of thumb used, extrinsic evidence and evidence for relative consistency are both extremely important. Before turning to these, I should mention that Reinhardt

[22]This particular voice of caution belonged to my thesis advisor, John Burgess.

[23]Moore [198?] points out that Gödel's attitude towards measurable cardinals had softened by 1966 when he thought their existence "followed from the existence of generalizations of Stone's representation theorem to Boolean algebras with operations on infinitely many elements".

[1974] has proved the existence of a measurable cardinal from a system which embodies what he claims to be a version of *reflection*. Martin, however, calls these "pseudo-reflection principles" [PSCN, p. 8] and Wang remarks that:

> ... reflection principles of diverse forms which are strong enough to justify measurable cardinals (by way of 1-extendible cardinals) no longer appear to be clearly implied by the iterative concept of set. [1974, p. 555]

I will take this up Reinhardt's ideas later, in [BAII, §VI], in connection with his closely-related defense of supercompact cardinals.

Given how seriously the Axiom of Measurable Cardinals has been pursued, it may seem surprising that the intrinsic and rule of thumb evidence is so scarce, but in this case the extrinsic evidence is extremely persuasive. The two most impressive consequences of the existence of measurable cardinals are that $V \neq L$ and that Σ^1_1 sets of reals are determined (Martin [1970]). I will discuss the second of these in [BAII, §V].

The first indication that the presence of measurable cardinals rules out $V = L$ came in Scott [1961], where he shows (using an ultrapower construction) that the measure on a measurable cardinal cannot be constructible. This is a welcome result ("so much the worse for the 'unnatural' constructible sets!" says Scott [1977, p. xii]), but perhaps not completely surprising given how complicated a measure must be. The nonconstructibility was brought closer to home by Rowbottom [1964] when he showed that the presence of a measurable cardinal guarantees a nonconstructible set of integers. Even further improvement came in Silver [1966] and Solovay [1967], where the nonconstructible set of integers is shown to be as simple as Δ^1_3. Notice that these results (as well as Martin's on the determinacy of analytic sets) confirm Gödel's prediction that the postulation of large cardinal numbers might yield new facts about sets of integers and reals.

Silver's model theoretic results show that $V \neq L$ can actually be derived from the existence of one particular Δ^1_3 set of integers, $0^\#$. $0^\#$ codes a set of formulas which in turn show how the constructible universe is generated by a proper class of order indiscernibles that contains all uncountable cardinals and more. Thus, not only does Silver's theory show *that* L goes wrong, it shows *how* L goes wrong: the process of taking only definable subsets at each stage yields a model statisfying ZFC at some countable stage, and all the further stages make no difference (this countable structure is an elementary substructure of L). The range of L's quantifiers is so deficient that L cannot tell one uncountable cardinal from another, or even from some countable ones. In purpler terms:

> L takes on the character of a very thin inner model indeed, bare ruined choirs appended to the slender life-giving spine which is the class of ordinals.

(Kanamori and Magidor [1978, p. 131]) The point is that $0^\#$ yields a rich explanatory theory of exactly where and why L goes wrong. Before Silver, many mathematicians believed that $V \neq L$, but after Silver they knew why.[24]

[24] Ironically, Silver's subsequent efforts have been to prove that measurable cardinals are inconsistent.

Thus the assumption that $0^{\#}$ exists presents a very attractive form of $V \neq L$. (Actually, the most attractive assumption is that $x^{\#}$ exists for every real number x, where $x^{\#}$ codes the indiscernible construction that shows $V \neq L[x]$.) This assumption turns out to be equivalent to a determinacy assumption and to an elementary embedding assumption (see [BAII, §§V and VI], respectively). This prompts Kanamori and Magidor [1978] to another application of diversity:

> [This] is a list of equivalences, much deeper than the confluence seen at weak compactness. (p. 140)

Thus the implication of the existence of the sharps provides a very appealing extrinsic support for the Axiom of Measurable Cardinals.

Another sort of extrinsic support comes from the fact that measurable cardinals allow patterns of results provable in ZFC alone to be extended. For example, as mentioned in II.3.1, Souslin's theorem that Σ^1_1 sets have the perfect subset property can be extended, in the presence of a measurable cardinal, to the Σ^1_2 sets (Solovay [1969]). The same goes for Lebesgue measurability and the Baire Property. (Borel determinacy and Σ^1_1 determinacy provide another example. See [BAII, §V].) Kanamori and Magidor emphasize yet another form of extrinsic support when they stress:

> ...the fruitfulness of the methods introduced in the context of large cardinal theory in leading to new 'standard' theorems of ZFC. (p. 105)

Many of these new methods (e.g. Silver forcing (p. 152), Ulam matrices (p. 162)), arose in studies of measurable cardinals. They also mention connections with other branches of mathematics (p. 109).

Finally, as promised, I should mention some of the evidence presented for the relative consistency of measurable cardinals. One idea is to show that various strong consequences of the Axiom of Measurable Cardinals are relatively consistent themselves; then we know at least that any inconsistency that follows from the existence of a measurable cardinal is not to be reached by those particular routes. So, for example, we know that $V \neq L$ is relatively consistent (Cohen [1966]), and furthermore, that the existence of a Δ^1_3 nonconstructible set of integers with properties much like those of $0^{\#}$ is relatively consistent (Jensen [1970] and Jensen and Solovay [1970]). Another line of argument runs that:

> ...some comfort can be gained from the fact that any number of attempts at showing that measurable cardinals do not exist have failed even though must cleverness was expended.

(Scott [1977, p. xii]). (See footnote 24.)

This is already a quite impressive list of extrinsic supports, but at least two more can be added. First, there is the inner model $L[U]$, where U is a normal measure on some uncountable cardinal κ. This model is the smallest in which κ is measurable, and it does not depend on the particular choice of U. Surprisingly, $L[U]$ shares many of the simplifying structural features of L: GCH is true, and there is a Δ^1_3 well-ordering of the reals (Silver [1971]). But this is only the beginning. Kunen's analysis of $L[U]$ via iterated ultrapowers [1970], and the work of Solovay and Dodd and

Jensen [1977] on the fine structure of $L[U]$ have revealed the "uniform generation and combinatorial clarity" of this inner model in considerable detail (Kanamori and Magidor [1978, p. 147]). The familiarity and depth of understanding provided by this inner model theory leads modern researchers to the view that:

> One of the main plausibility arguments for measurable cardinals is that they have natural inner models.

(Kanamori and Magidor [1978, p. 147]) The canonical inner model makes measurable cardinals much less mysterious.

Second, concentration on the nontrivial elementary embedding of V into a transitive model M that is provided by Scott's ultrapower construction revealed that the existence of such an embedding is in fact equivalent to the existence of a measurable cardinal. (The first ordinal moved by such an embedding must be measurable.)

> Thus, the really structural characterization of measurable cardinals in set theory emerged.

(Kanamori and Magidor [1978, p. 110]) Though this is not quite the *inexhaustibility* characterization that was hoped for, it is simple and basic, and it does lead to many fruitful generalizations (see [BAII, §VI]). If the definition in terms of measures or ultrafilters had once seemed unmotivated, the connection with elementary embedding via ultrapowers revealed its true nature. Furthermore, elementary embedding cardinals are more amenable to the rule of thumb justifications that elude measurable cardinals in their original guise (see [BAII, §VI]).

Given this wide range of support for the Axiom of Measurable Cardinals, it is perhaps not surprising that proof from that axiom, at least among members of the Cabal group, has come to be treated as tantamount to proof outright. Here we have, as Gödel predicted, an axiom so rich in extrinsic supports that:

> ...whether or not [it is] intrinsically necessary, [it is] accepted at least in the same sense as any well-established physical theory. [1947/64, p. 477]

Unfortunately, for all that, it cannot answer the question we had hoped it would. Levy and Solovay [1967] have shown that measurable cardinals, and indeed all large cardinals of the sort developed so far, are preserved under most forcing extensions, and thus, that they can be shown to be relatively consistent with both the continuum hypothesis and its negation.

In [BAII], I will consider axiom candidates of a completely different sort—hypotheses on determinacy—along with the larger large cardinals.

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