Notes on 4-coloring the 17 by 17 grid

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August 5, 2009

1 For large color classes, ≤ 5 in each row, column

A color class is large if it contains at least 73 points. We know that in all colorings of the 17 by 17 array, there is at least one large color class (pigeonhole principle).

I take columns to be the main structural component of a color class. That is, in my mind a color class is really a set of sets, where each set represents a column, and the letters in that set correspond to the rows that that color class occupies in that column. Since there are 17 rows, I use the letters A-Q to represent them (under some fixed permutation). Having two columns "overlap" as sets isn't a problem, but to avoid rectangles we need that no two columns overlap in two or more places, or that the pairs of rows contained in each column are all distinct from each other. There are only 136 total pairs of rows, so any color class that encompasses more than 136 pairs of rows has to be impossible. For example, if we had 5 in every column, each column would have 10 pairs of rows making 170 pairs of rows total. Since there are only 136 pairs of rows, somewhere two columns share a pair of rows, which corresponds to a rectangle, and so this isn't a valid color class.

1.1 Proof

List the number of points in each column in a fixed large color class, in order from largest to smallest. We know that the first 5 columns must sum to no more than 27, otherwise there would be two columns that overlap at two rows. Since the whole thing sums to at least 73, that means that the first 5 columns must all be at least 4. Thus, the first 4 columns sum to less than 23, the first 3 to less than 19, and the first two to less than 15. Moreover, we know that the first five columns cannot sum to less than 25 - if they are all five (or more), we have at least 25. If some are 4s, then the rest of the numbers must be ≤ 4, and we cannot achieve 73. Here are ALL of the possibilities for the first 5 columns:

Sum to 27:

11	4	4	4	4
10	5	4	4	4
9	6	4	4	4
8	7	4	4	4
9	5	5	4	4
8	6	5	4	4
7	7	5	4	4
7	6	6	4	4
8	5	5	5	4
7	6	5	5	4
6	6	6	5	4
7	5	5	5	5
6	6	5	5	5

Sum to 26:

10	4	4	4	4
9	5	4	4	4
8	6	4	4	4
7	7	4	4	4
8	5	5	4	4
7	6	5	4	4
6	6	6	4	4
7	5	5	5	4
6	6	5	5	4
6	5	5	5	5

Sum to 25:

9	4	4	4	4
8	5	4	4	4
7	6	4	4	4
7	5	5	4	4
6	6	5	4	4
6	5	5	5	4
5	5	5	5	5

Now suppose that the first two columns sum to 13 or more. If it's spread out evenly, we will need at least 13 more columns to be 4 or more (from the fact that the max sum of the first two columns is 15). The first two columns can either share a single row, leaving 5 unused row, or they can be disjoint, leaving 4. In either case, we will combine pairs of unused rows with pairs from the first and second column, to form as many groups of 4 as possible. If we have 5 unused rows, we can form only $\binom{5}{2} = 10$ pairs, and so this fails since we haven't found 13. If we have 4 unused rows, we can only form $\binom{4}{2} = 6$ pairs, and so this fails as well.

With similar logic, we can handle the case where the first two columns sum to 12 exactly. We would need one column of 5 and then 14 columns of 4. If the first two columns don't overlap in any letters we only have 5 unused row, and can only hope to get 10 pairs, which isn't enough. If they do overlap they would leave 6 rows unused, but to get a column of 5 we would need a group of 3 unused rows, and so we would actually need 17 pairs, when we only have 15.

We've assumed that things are spread out evenly - that we have lots of 4s, rather than lots of 5s and 3's (or 2's and 6's, etc.). The reason this works is that a 5 and a 3 use 13 pairs, while two 4s only use 12. In our case, we're looking actually at pairs vs. trios and singles, but the same principle applies: two pairs are, well, two pairs, whereas a trio and a single cover three pairs. All of this is really from the fact that the function $\binom{x}{2}$ is concave, so we know in general that as we move away form the most evenly spread out possibility, the number of pairs we will use goes up. Since we're showing that we'd be taking away too many pairs than we have, it suffices to work in the case where we're taking the least number of pairs possible.

Here are our remaining possibilities:

Sum to 27:	Sum to 26:	Sum to 25:
∅	6 5 5 5 5	6 5 5 5 4 5 5 5 5 5

Now assume that the first 4 columns are 6,5,5,5. These 4 columns could be arranged in the following ways:

Case 1) each column overlaps with each other column at a distinct row (i.e. no 3 columns overlap at the same row) - uses only the first 15 rows.

Case 2) the three columns of 5 overlap at a single row, each overlaps with the column of length 6 at a distinct row - uses 16 of the rows

Case 3) the column of length 6 and two of the columns of length 5 overlap at a single row, the last column of length 5 overlaps with each of the rest at different rows - uses 16 of the rows

Case 4) There are pairs of columns that do not overlap in any rows.

Note - this is exhaustive since we cannot have all four columns overlap in a single row, and we cannot have two groups of 3 columns overlapping in the same row.

Case 1				Case 2				Case 3			
A	A			A	A			A	A	A	
B		B		B		B		B			B
C			C	C			C	C			
D				D				D			
E				E				E			
F				F				F			
	G	G			G	G	G		G		G
	H		H		H				H		
	I				I				I		
	J				J				J		
		K	K			K				K	K
		L				L				L	
		M				M				M	
			N				N			N	
			O				O				O
							P				P

In all cases we have used 21 points, and so to achieve 73 points total we need at least 52 more, with columns no greater than 5 (since we have them ordered already from largest to smallest). This means that for some constant i, $0 \leq i \leq 6$, we will have $13-2i$ columns of 4, i columns of 5 and i columns of 3. $(13-2i)4+(5+3)i = 13(4) = 52$.

Case 1: We have two unused rows left over - call them P and Q. Each column with five overlaps distinctly with the other 3 columns, so each column of 5 only has two unused elements. We can make 4 pairs of 3 from these, but it happens that to complete them to be groups of 4 with the elements of the column of 6 we would need 4 elements of the group of 6 that haven't already been grouped with any of those rows, which we don't have. So we can make at most 3 groups of 4 without using P or Q. We can have one column of size 5 that contains both P and Q - other than that, there cannot be a column of size five apart from one of the original columns of size 4 with an extra P or Q (so there cannot be more than 4 columns of size 5). In addition, since there is a group of six, we have further limitations on how many columns P and Q can each be in.

If P and Q are not in a column of size 5 together we have the following options for P (or equivalently, Q): 4 columns of size 4, 1 column of size 5 and 3 of size 4, 2 columns of size 5 and one of size 4, 3 columns of size 5. Put together, we have the following possibilites:

original:	with P:	with Q:	max number used	
3 size 4	4 size 4	4 size 4	$11 \cdot 4 + 2 \cdot 3$	$= 50$
2 size 4	1 size 5, 3 size 4	4 size 4	$1 \cdot 5 + 9 \cdot 4 + 3 \cdot 3$	$= 50$
1 size 4	1 size 5, 3 size 4	1 size 5, 3 size 4	$2 \cdot 5 + 7 \cdot 4 + 4 \cdot 3$	$= 50$
1 size 4	2 size 5, 1 size 4	4 size 4	$2 \cdot 5 + 6 \cdot 4 + 5 \cdot 3$	$= 49$
0 size 4	2 size 5, 1 size 4	1 size 5, 3 size 4	$3 \cdot 5 + 4 \cdot 4 + 6 \cdot 3$	$= 49$
0 size 4	3 size 5	4 size 4	$3 \cdot 5 + 4 \cdot 4 + 6 \cdot 3$	$= 49$

If P and Q are in a column of size 5 together we have the following options for P (or equivalently Q) in addition to the column of size 5 they share: 3 columns of size 4, 1 column of size 5 and 2 columns of size 4, 2 columns of size 5

original:	with P:	with Q:	max number used	
3 size 4	3 size 4	3 size 4	$1 \cdot 5 + 9 \cdot 4 + 3 \cdot 3$	$= 50$
2 size 4	1 size 5, 2 size 4	3 size 4	$2 \cdot 5 + 7 \cdot 4 + 4 \cdot 3$	$= 50$
1 size 4	1 size 5, 2 size 4	1 size 5, 2 size 4	$3 \cdot 5 + 5 \cdot 4 + 5 \cdot 3$	$= 50$
1 size 4	2 size 5	3 size 4	$3 \cdot 5 + 4 \cdot 4 + 6 \cdot 3$	$= 49$
0 size 4	2 size 5	1 size 5, 2 size 4	$4 \cdot 5 + 2 \cdot 4 + 7 \cdot 3$	$= 49$

Since in none of these case we have achieved 52 points, in this configuration 6,5,5,5 is impossible.

Case 2: Call the unused row Q. G cannot be in any group of 4 - the only letters G hasn't already been grouped with are D, E, F and Q. A can only be in a group of 4 that contains Q (and no group of 5), so A can only be in one more group of 4 and it will miss out on being paired with at least 4 letters. Similarly for B and C. This means the total number of pairs of rows we cold hope to get is $136 - 16 = 120$. But in order to complete the color class the number of pairs we must use is at least

$$\binom{6}{2} + 3\binom{5}{2} + (13 - 2i)\binom{4}{2} + i\binom{5}{2} + i\binom{3}{2}$$
$$= (15 + 30 + 6(13)) + i(10 + 3 - 12) = 123 + i \geq 123.$$

Again, we use the most evenly distributed case above, because it is the smallest number of pairs we could hope to use (in particular, we are excluding the case where we have columns of size 2 or smaller). Contradiction, therefore 6,5,5,5 is not possible in this configuration.

Case 3: Call the unused row Q. There can be at most 6 groups of 4 without using Q (and none of them can be extended to 5s), so that only uses up 24 points and 6 columns, leaving 28 points and 7 columns to be in columns with q or in columns of size ≤ 3. Q itself can only be with each of the other 16 rows once, and there is a group of 6, which further limits the possibilities for Q. Our options for the remaining 7 columns are:

with Q:	max number used	
5 columns of size 4	$5 \cdot 4 + 2 \cdot 3$	$= 26$
2 columns of size 5 and and 2 of size 4	$2 \cdot 5 + 2 \cdot 4 + 3 \cdot 3$	$= 27$
3 columns of size 5	$3 \cdot 5 + 4 \cdot 3$	$= 27$

Since we cannot use 28 points, we cannot achieve 6,5,5,5 in this configuration.

Case 4: Intuitively, this ought to be worse than if we constructively used the overlap (such as gathered it at a single row). I think there ought to be a general argument that handles this case. But for the moment, see the appendix for (more) case analysis.

Therefore, the maximum number in any column of a large color class is 5. Since the choice of columns over rows was arbitrary, the max number in any row is 5 as well.

1.2 Maximum size of largest color class

When we switch two 4 columns to be a 3 column and a 5 column, we end up using 13 pairs instead of 12. If we have 2 in any column, and we switch a 4 and a 2 to two 3's we go down as well, or a 5 and a 2 to a 4 and a 3, we're better off as well. So the least number of pairs we'll use, for a given number of points, is when we use all 4s and 5s. This lets us solve for the maximum number of points possible. Let F be the number of fives, and R be the number of fours. We have the following:

$$F + R = 17 \implies R = 17 - F$$
$$10F + 6R \leq 136$$
$$10F + 6(17 - F) \leq 136$$
$$4F \leq 136 - 102 = 34$$
$$F \leq 34/4 = 8.5$$

Therefore, the maximum number of points that can be achieved is when there are 8 5's and 9 4's, or a total of $40 + 36 = 76$ points. Note that this is a pretty small window - a large color class must have at least 73 points.

1.3 74 is achievable

Consider first the following non-example:

B	F	J	N	B	C	D	E	B	C	D	E	B	C	D	E	B	C	D	E
C	G	K	O	F	G	H	I	G	F	I	H	H	I	F	G	I	H	G	F
D	H	L	P	J	K	L	M	L	M	J	K	M	L	K	J	K	J	M	L
E	I	M	Q	N	O	P	Q	Q	P	O	N	O	N	Q	P	P	Q	N	O
A	A	A	A																

If we separate out all of the columns containing Q, we have:

B	F	J	B	C	D	C	D	E	B	C	E	B	D	E	N	E	B	D	C
C	G	K	F	G	H	F	I	H	H	I	G	I	G	F	O	I	G	F	H
D	H	L	J	K	L	M	J	K	M	L	J	K	M	L	P	M	L	K	K
E	I	M	N	O	P	P	O	N	O	N	P	P	N	O	Q	Q	Q	Q	Q
A	A	A													A				

From here we can see that If we get rid of the last three columns (corresponding to the sets BGLQ, DFKQ, CHKQ), and remove Q from the set NOPQA, we can add Q to some of the other columns, to make more columns of length 5:

B	F	J	N	B	C	D	E	C	D	E	B	C	E	B	D	E
C	G	K	O	F	G	H	I	F	I	H	H	I	G	I	G	F
D	H	L	P	J	K	L	M	M	J	K	M	L	J	K	M	L
E	I	M	A	N	O	P	Q	P	O	N	O	N	P	P	N	O
A	A	A		Q	Q	Q										

We now have 17 columns, 6 of length 5 and 11 of length 4: $6 \cdot 5 + 11 \cdot 4 = 74$.

1.4 Is 75 achievable?

If 75 is achievable, we have to have (at least) the 7 largest columns be 5, and in addition we will need to use at least $70 + 60 = 130$ of the pairs of rows.

The columns must be in one of the following configurations:

Note - all columns overlap in distinct rows with all other columns - not actually possible! Any column of 5 must overlap with 2 other columns in the same row, or be disjoint from one of the other columns. In addition, there must be at least one row with three columns present. See table for illustration of the cases:

1) One group of 3 columns overlap in a single row, all other columns overlap in distinct rows (except for two pairs of columns that are disjoint). Impossible pairs: AN, AO, AP, AQ, OP, NQ. Argument - BQ is impossible as well -FAIL

2) Two disjoint groups of 3 columns overlap in a single row, all other columns overlap in distinct rows, remaining column is disjoint from one other column. Impossible pairs: AO, AP, AQ, BF, BJ, BN, AB - FAIL

3) Two overlapping groups of 3 columns overlap in a single row, all other columns overlap in distinct rows (remaining two columns disjoint from each other). Impossible pairs: AN,AO,AP,AQ,BE,BF,BI,BJ - FAIL.

4) A chain of 3 groups of 3 columns overlap in a single row. Impossible Pairs: AC,AO,AP,AQ, CD,CH,CN - FAIL

5) A star of 3 groups of 3 columns overlap in a single row. Impossible Pairs: AN,AO,AP,AQ,BH,BI,BL,BM - FAIL

Note - triangles of hyperedges are impossible. Groups of 4 hyperedges: cannot contain triangles! "Square" shape happens to also be impossible. Cannot find any other configurations...formal argument to prove it?

Note - the result that 75 is not achievable also shows that nothing beginning with 7 columns of 5 is achievable. This gives us that 74 is the max AND that 74 must be of the form $\underbrace{5,\ldots,5}_{6},\underbrace{4,\ldots,4}_{11}$. It does not say that the construction above is the only one possible, however.

2 Implications

2.1 The smallest color class

Since the max achievable is 74, the smallest color class is still of size $289-3(74) = 67$. This lets us say some interesting things about how a color class of this size could be constructed. As we argued before, the five largest columns must sum to ≤ 27, and they must all be larger than 4. So the 4 largest columns sum to be ≤ 23, the three largest to be ≤ 19, and the two largest to be ≤ 15. Thus we have the same list of possible beginnings as before, with some additions:

case 1						
A	A	A				
B			B			
C				C		
D					D	
E						E
	F		F			
	G			G		
	H				H	
	I					I
		J	J			
		K		K		
		L			L	
		M				M
			N		N	
			O			O
				P	P	
				Q		Q

case 2						
A	A	A				
			B	B	B	
C			C			
D				D		
E					E	
F						F
	G		G			
	H			H		
	I				I	
	J					J
		K	K			
		L		L		
		M			M	
		N				N
			O			O
				P		P
					Q	

case 3						
A	A	A				
		B	B	B		
C			C			
D				D		
E					E	
F						F
	G		G			
	H			H		
	I				I	
	J					J
		K			K	
		L				L
		M				
			N		N	
			O			O
				P	P	
				Q		Q

case 4						
A	A	A				
		B	B	B		
				C	C	C
D			D			
E				E		
F					F	
G						G
	H		H			
	I			I		
	J				J	
	K					K
		L			L	
		M				M
		N				
			O		O	
			P			P
				Q		

case 5						
A	A	A				
B			B	B		
C					C	C
D						
E						
	F		F			
	G			G		
	H				H	
	I					I
		J	J			
		K		K		
		L			L	
		M				M
			N		N	
			O			O
				P	P	
				Q		Q

Table 1: Illustrations for the various cases in trying to achieve 75

Sum to 27:

11	4	4	4	4
10	5	4	4	4
9	6	4	4	4
8	7	4	4	4
9	5	5	4	4
8	6	5	4	4
7	7	5	4	4
7	6	6	4	4
8	5	5	5	4
7	6	5	5	4
6	6	6	5	4
7	5	5	5	5
6	6	5	5	5

Sum to 26:

10	4	4	4	4
9	5	4	4	4
8	6	4	4	4
7	7	4	4	4
8	5	5	4	4
7	6	5	4	4
6	6	6	4	4
7	5	5	5	4
6	6	5	5	4
6	5	5	5	5

Sum to 25

9	4	4	4	4
8	5	4	4	4
7	6	4	4	4
7	5	5	4	4
6	6	5	4	4
6	5	5	5	4
5	5	5	5	5

Sum to 24:

8	4	4	4	4
7	5	4	4	4
6	6	4	4	4
6	5	5	4	4
5	5	5	5	4

Sum to 23:

7	4	4	4	4
6	5	4	4	4
5	5	5	4	4

Sum to 22:

6	4	4	4	4
5	5	4	4	4

Sum to 21

5	4	4	4	4

Sum to 20:

4	4	4	4	4

If the first two columns sum to be at least 14, we have at most 4 unused elements left over, and we need at least 7 more 4s. So we'd need 7 pairs from these 4 unused elments, but $\binom{4}{2}$ is only 6. If we tried to make some of those into 5s, we'd run into trouble - we can only get one group of three from the set of 4 unused elements, and then we'd only have 3 pairs left over - 14+5+12 + 3(11) = 64 - FAIL

If the first column is 9 (and we don't sum to 14 or more), we have 8 unused rows after the first column, and we need at least 10 groups of 4 to achieve 67. So from these 8 unused rows we need to find 10 columns with three each (this assumes that we won't have any problems adding a 4th from the original group of 9). But there are only $\binom{8}{2} = 28$ pairs of rows among the 8 unused rows, and if were to find 10 columns with three that don't form a rectangle we would use $10 \cdot \binom{3}{2} = 30$ pairs, which is clearly impossible.

This already reduces the possibilities somewhat. Moreover, when the first 4 columns sum to 23 (note that they cannot sum to more) we will use all possible rows with them, and it becomes very clear how many columns with 4 we could possibly hope for (multiply the two smallest numbers of open rows). We need at least 5 groups of 4 to achieve 67.

7,6,6,4 - FAIL			
A	A		
B		B	
C			C
D			
E			
F			
G			
	H	H	
	I		I
	J		
	K		
	L		
		M	M
		N	
		O	
		P	
			Q

8,5,5,5 - FAIL			
A	A		
B		B	
C			C
D			
E			
F			
G			
H			
	I	I	
	J		J
	K		
	L		
		M	M
		N	
		O	
			P
			Q

7,6,5,5 - FAIL			
A	A		
B		B	
C			C
D			
E			
F			
G			
	H	H	
	I		
	J		
	K		
	L		
		M	M
		N	
		O	
			P
			Q

6,6,6,5 - PASS			
A	A		
B		B	
C			C
D			
E			
F			
	G	G	
	H		H
	I		
	J		
	K		
		L	L
		M	
		N	
		O	
			P
			Q

Note that this doesn't mean that 6,6,6,5 is necessarily achievable, just that it doesn't fail this test.

To rule out the rest of the ones with 8 in the first column, consider if we have 8 and 11 4's, and 5 3's. There are 9 unused rows, from these 9 rows we will need 11 groups of 3 and 5 groups of 2, using a total of $33 + 5 = 38$ pairs of rows. But $\binom{9}{2}$ is only 36, and so this fails. Similarly, to get 1 group of 4, 9 groups of 3 and 6 groups of 2 we'd use $6 + 27 + 6 = 39$, and to get 2 groups of 4, 7 groups of 3 and 7 groups 2 we'd use $12 + 21 + 7 = 40$, so these are impossible.

At this point, our options for the first 5 columns are:

Sum to 27:

6	6	6	5	4
7	5	5	5	5
6	6	5	5	5

Sum to 26:

7	6	5	4	4
6	6	6	4	4
7	5	5	5	4
6	6	5	5	4
6	5	5	5	5

Sum to 25

7	6	4	4	4
7	5	5	4	4
6	6	5	4	4
6	5	5	5	4
5	5	5	5	5

Sum to 24:

7	5	4	4	4
6	6	4	4	4
6	5	5	4	4
5	5	5	5	4

Sum to 23:

7	4	4	4	4
6	5	4	4	4
5	5	5	4	4

Sum to 22:

6	4	4	4	4
5	5	4	4	4

Sum to 21

5	4	4	4	4

Sum to 20:

4	4	4	4	4

So we know that no color class contains more than seven in any column or (by reflection) any row. In fact, we know more about the ones where we have 67 exactly - it must fit in with 3 74s (this is the only way it could occur), and each 74 either has 4 or 5 in every column. So in fact we know that the largest column the 67 could have is 5, corresponding to a column in which all the 74s have only 4.

2.2 Filling in the interval 68-74

Claim: 70 or more cannot have 7 (or more) in any row or column

Options for the first 5 columns for ≥ 70 - this is a subset of the options for the smallest color class, above:

Sum to 27:

6	6	6	5	4
7	5	5	5	5
6	6	5	5	5

Sum to 26:

7	6	5	4	4
6	6	6	4	4
7	5	5	5	4
6	6	5	5	4
6	5	5	5	5

Sum to 25

7	6	4	4	4
7	5	5	4	4
6	6	5	4	4
6	5	5	5	4
5	5	5	5	5

Sum to 24:

7	5	4	4	4
6	6	4	4	4
6	5	5	4	4
5	5	5	5	4

Sum to 23:

7	4	4	4	4
6	5	4	4	4
5	5	5	4	4

Sum to 22:

6	4	4	4	4
5	5	4	4	4

If the first column is 7, that leaves 10 unused rows and only 45 pairs of unused rows. We need to have i 5s, $15 - 2i$ 4s and $i + 1$ 3s. This means, that from the 10 unused rows we will take this many pairs: $i(6) + (15 - 2i)3 + i + 1 = i + 46 > 45$.

69 and 68, however, are both achievable with 7 in one column. For 69, we will take $6i + (14 - 2i)3 + i + 2 = 44 + i$ pairs, so we might expect solutions for i=0 and i=1. For 68, we will take $6i + (13 - 2i)3 + i + 3 = 42 + i$ pairs, so we might expect solutions for $0 \leq i \leq 3$. In fact, we can't quite do this well - 69 is achievable with i=1 but not i=0. 68 is achievable for all values of i. This has to do with trying to find lots of columns of height 3, which is basically just a smaller version of this same problem. Proof - see the appendix.

In a legal coloring of the whole array, there can't be more than one color class with 67, 68 or 69 points. So if 68 has a seven in a legal coloring, then there must be an instance of 73 with a 2, but that 73 would need 7 fives, and we know (from the fact that 75 isn't achievable) that that's impossible, so 68 cannot have a seven in such an overall coloring. 69, however, could have a 7 in a legal coloring, since it could be matched with 73, 73 and 74. If both 73s have a column with 3, and those columns happen to overlap, and correspond to a column of 74 that only has 4 elements, the 69 would have to have 7.

Claim: 72 cannot have a 6 in any row or column:

Proof is long and messy - see appendix.

3 How unique is the construction of 74?

We know that for 74 to exist it must be of the form $\underbrace{5, \ldots, 5}_{6}, \underbrace{4, \ldots, 4}_{11}$. This means that it will use $60 + 66 = 126$ of the potential 136 pairs. Following the argument that 75 is not achievable, we have the following possibilities for the first 6 columns:

1) One group of 3 columns of 5 overlap in a single row, all other columns of 5 overlap in distinct rows. We need 11 groups of 4, which we almost have because Q can be in as many as 5 groups of 4, and we have the openings PM, PJ, OM, OK, NM, NL. It turns out that it's impossible to complete all 6 of these openings, however, so we fail (see appendix).

2) Two disjoint groups of 3 columns of 5 overlap in a single row, all other columns of 5 overlap in distinct rows - we know that this works (this is the example shown).

3) Two overlapping groups of 3 columns of 5 overlap in a single row, all other columns overlap in distinct rows. Fails, in a complicated way (see appendix).

4) Every column overlaps every other column in distinct rows. This uses only 15 of the 17 rows, leaving the last two completely unused (P and Q). However, there are no groups of 4 that can be made without P or Q, and we cannot get 11 groups of 4 involving P and or Q. Fails.

case 1

A	A	A			
B			B		
C				C	
D					D
E					
	F		F		
	G			G	
	H				H
	I				
		J	J		
		K		K	
		L			L
		M			
			N	N	
			O		O
				P	P

case 2

A	A	A			
			B	B	B
C			C		
D				D	
E					E
F					
	G		G		
	H			H	
	I				I
	J				
		K	K		
		L		L	
		M			M
		N			
			O		
				P	
					Q

case 3

A	A	A			
		B	B	B	
C			C		
D				D	
E					E
F					
	G		G		
	H			H	
	I				I
	J				
		K			K
		L			
		M			
			N		N
			O		
				P	P
				Q	

case 4

A	A				
B		B			
C			C		
D				D	
E					E
	F	F			
	G		G		
	H			H	
	I				I
		J	J		
		K		K	
		L			L
			M	M	
			N		N
				O	O

case 5

A	A	A			
B			B		
C				C	
D					
E					
	F		F		
	G			G	
	H				H
	I				
		J	J		
		K		K	
		L			L
		M			
			N	N	
			O		O
				P	P
					Q

case 6

A	A	A			
B			B		
C				C	
D					D
E					
	F		F		
	G			G	
	H				H
	I				
		J	J		
		K		K	
		L			L
		M			
			N	N	
			O		
				P	P
					Q

5) One group of 3 columns overlap in a single row, one other column is disjoint from that group of three. Our possible pairs to begin groups of 4 are NQ, NM, NL, OM, OK, PM, PJ, QM, QJ, QK, which is not enough. Fails.

6) One group of 3 columns overlap in a single row, one pair of columns not in that group are disjoint from each other. Our options for possible groups of 4 are PO, QO, QN, PM, PJ, QM, QK, QJ, NM, NL, OK, OL and OM. However P can only be matched with F, I, J and M, and so P can be in either both of PM and PJ, or it can be in one of them and we keep PO, but we have to lose one pair with P. Similarly, we lose one for N, O and Q, and so we end up with too few pairs.

There are three other cases, where we let columns be disjoint (without gathering any 3 columns at a single row). Again, I feel like there's a general argument to be made here, but for the moment these are handled in the appendix.

So an achievable 74 has two disjoint groups of 3 columns overlapping in a single row:

A	A	A			
			B	B	B
C			C		
D				D	
E					E
F					
	G		G		
	H			H	
	I				I
	J				
		K	K		
		L		L	
		M			M
		N			
			O		
				P	
					Q

Our possibilities for more groups of 4 are AOPQ, BFJN, and (working from the bottom) ones beginning PN,PM,PK, QN, QL, QK, ON, OM, OL. Note that if we try to do something clever with adding the pairs OP, OQ and PQ we end up having to take corresponding numbers of pairs out, since there are only so many other letters free, plus we lose the group AOPQ. So this is in fact the only possible list of opening pairs that has the potential to be completed. When we look at the possibilities for completing those groups of 4, we get this:

P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
GIJ	GJ	IJ	GHJ	GJ	HJ	HIJ	HJ	IJ
CEF	CF	EF	CDF	CF	DF	DEF	DF	EF

As soon as we assign the third letter to the pair PN, however, we nail down a lot of information:

P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
G	J	I	GHJ	GJ	HJ	HIJ	HJ	IJ
E	C	F	CDF	CF	DF	DEF	DF	EF

P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
I	G	J	GHJ	GJ	HJ	HIJ	HJ	IJ
C	F	E	CDF	CF	DF	DEF	DF	EF

This last one turns out to be impossible:

P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
J	G	I	GHJ	GJ	HJ	HIJ	HJ	IJ
CEF	?F	?F	CDF	CF	DF	DEF	DF	EF

Now that the ones with P have been decided, they give us information about O and Q. For example, in the first case, M can no longer be paired with J, making us pick OMH instead of OMJ. N and G can no longer be paired, making the only way Q will be paired with G is if we have QLG. And so we can fill out the rest of the columns:

P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
G	J	I	H	G	J	I	H	J
E	C	F	C	F	D	D	F	E

P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
I	G	J	G	J	H	H	J	I
C	F	E	D	C	F	E	D	F

From this we see that there is really only one degree of freedom - that is, only 2 distinct instances of 74, up to permutation of letters.

3.1 Comparing the two different 74s

This is assuming certain row and column permutations. Letters are points common to both color classes, copyright signs are points in one class and the ✠ are points in the other.

A	A	A	A												
				B	B	B	B								
C				C								©	✠		✠	©
D					D				✠	©				©		✠
E						E		©		✠	✠		©			
F							F	✠	©		©	✠		✠	©	
	G			G								✠	©		©	✠
	H				H				©	✠				✠		©
	I					I		✠		©	©		✠			
	J						J	©	✠		✠	©		©	✠	
		K		K							K			K		
		L			L			L							L	
		M				M			M			M				
		N					N			N			N			N
			O	O				O	O	O						
			P		P						P	P	P			
			Q			Q								Q	Q	Q

3.2 How unique is the construction of 73?

73 must either be of the form $\underbrace{5,\ldots,5}_{5},\underbrace{4,\ldots 4}_{12}$ or $\underbrace{5,\ldots,5}_{6},\underbrace{4,\ldots 4}_{10},3$, and both are achievable by removing a point from 74. But are there constructions of 73 that aren't subsets of 74?

3.2.1 6 5's, 10 4's, 1 3

We have the same 9 cases for the position of the first 6 columns with 5 in them as in the proof that 74 is (almost) unique, and so we use the exact same table (see above):

1) One group of 3 columns overlap in a single row, all other columns overlap in distinct rows. We need 10 groups of 4 and one group of 3, which we almost have because Q can be in as many as 5 groups of 4, and we have the openings PM, PJ, OM, OK, NM, NL. It turns out that not only is it impossible to complete all 6 of these openings, however, we can only complete at most 3 of them, so we fail by the same argument as before (see appendix for the original argument, in detail).

2) Two disjoint groups of 3 columns overlap in a single row, all other columns overlap in distinct rows, remaining column is disjoint from one other column. We know that this works (this is the example shown).

3) Two overlapping groups of 3 columns overlap in a single row, all other columns overlap in distinct rows. Fails for the same reasons as before, but you have to take them one step further (see appendix).

4) Every column overlaps every other column in distinct rows. This uses only 15 of the 17 rows, leaving the last two completely unused (P and Q). Turns out it's impossible to fill all 5 for both P AND Q - tricky to show, though.

5) One group of 3 columns overlap in a single row, one other column is disjoint from that group of three. Our possible pairs to begin groups of 4 are NQ, NM, NL, OM, OL, OK, PM, PL, PJ, QM, QL, QJ, QK.However we can only complete 3 Qs, 2Ps, 2N's and 2Os - we need to remove too many of these pairs to have enough survive. Fails.

6) One group of 3 columns overlap in a single row, one pair of columns not in that group are disjoint from eachother. Our options for possible groups of 4 are PO, QP, QN, PM, PL, PJ, QM, QK, QJ, NM, NL, OK, OM. However we can only complete 3 Qs, 3 Ps, 2Ns and 2Os, leaving us just ten potential groups of 4. It is actually impossible to add letters from the set { FGHI } to them to even get up to groups of 3, so this fails.

Cases 7, 8 and 9 are treated in the appendix with the original cases 7 8 and 9 - they generalize directly.

So far so good - we know that the columns of 5 must be in the right positions.

A	A	A			
			B	B	B
C			C		
D				D	
E					E
F					
	G		G		
	H			H	
	I				I
	J				
		K	K		
		L		L	
		M			M
		N			
			O		
				P	
					Q

If we assume that we have AOPQ as a group of 4, we can continue the argument for having only two kinds of 74. We can have one group of 3 or 4 from the set {B,F,J,N}, and then all the other groups of 3 or 4 must come from {O,P,Q}, {K,L,M,N}, {G,H,I,J} and {C,D,E,F}. If we assume that we have only a group of 3 from BFJN, then the argument follows directly - there are only 2 ways of completing the groups of 4, and we can always extend back to a 74 (we didn't use that fact in filling out the rest of the groups of 4). If not, then one of {O,P,Q} is in a group of 3. By symmetry, assume it's O. Then we get these two tables still:

P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
G	J	I	HJ	GJ	HJ	HIJ	H	IJ
E	C	F	CDF	CF	D	DF	DF	EF

P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
I	G	J	GHJ	GJ	H	HJ	HJ	IJ
C	F	E	DF	CF	DF	DEF	D	EF

Q is forced in both cases, which gives us even more information about the possibilities for O:

P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
G	J	I	H	G	J	IJ	H	J
E	C	F	C	F	D	DF	F	E
P	P	P	Q	Q	Q	O	O	O
N	M	K	N	L	K	N	M	L
I	G	J	G	J	H	HJ	J	I
C	F	E	D	C	F	EF	D	F

In both cases, our only options for completing all groups of 4 with O are determined. We may think that we can do something sneaky, like in the first case, we complete the first column of O as ONJD (which is not legal when we assume that we need all groups of 4 with O), and then drop the J from OLJE to make a group of 3. But when we add in the extra information we're assuming, that BFJN is a group as well, we see that we are ironed out - assuming that we need all the groups of 4 from P and Q, and BFJN, leaves us no freedom with the groups of 4 from O.

If we don't assume AOPQ as a group of 4, but just OPQ, the argument above follows just as easily. We really need AOPQ just to eliminate the possibility of groups using two elements from the set {O,P,Q}. So instead if we take AOP (equivalent to AOQ or APQ by symmetry), we have the possibility that we have any nine or ten of the following 11 openings: ON, OM, OL, PN, PM, PK, QN, QL, QK, QO, QP (and perhaps BFJN if we can only get 9 of them, but let's not assume that until we have to). We know that A and B can't be involved in any of these openings, so they have to be completed with letters from {CDEF} {GHIJ} {KLMN}. But Q can't be paired with E, I or M, so there are only nine letters left for Q to go with, which can only make 4 more pairs. So Q cannot be in 5 openings, and we need to get rid of one with Q in it. If it's either QO or QP (assume QO arbitrarily), we still have QN, QL, QK, which each need a pair from {CDF} and {GHJ} - this uses up both of those sets, so QP cannot be completed. If we drop both QO and QP we're back to normal.

Therefore, it must be one of QN, QK, QL that gets dropped if we want a new answer. QL and QK are equivalent, QN is not (unless we assume BFNJ), so let's first see what happens when we drop QL. We have the following pairs to be completed: QO, QP, QK and QN, and L is now free - assume we put L with QO - note that this eats up the opening OL as well, so we would really need to complete all the ones with Q for this to work. We have: QOL, QP{GJ}, QK{HJ}, QN{GHJ}, and all 4 of these need a letter from the set {CDF} - this is clearly impossible, and so this fails. If we dropped QN instead, we would be in the same position (we only used the size of the set {CDF}, not its interactions with the set {KLN}).

Therefore, every 73 with 6 columns of 5 is the subset of a 74.

3.2.2 5 5's, 12 4's

For 5 columns of 5, there aren't that many options.

1). All 5 overlap distinctly - no two columns disjoint. We can only get one group of 4 not including P or Q (OJIA, NKIA, MLIA or ONL(?), but not more than one of those). P and Q can only contribute 5 groups of 4 each, and so we don't have enough. Fails.

2). Columns 1 and 2 disjoint. Without Q, our options for groups of 4 begin PO, PK, PM, OL, OM, NM. Q can only be in 5 groups of 4, and so we can only ever hope for 11.

3) Columns 1 and 2 and 2 and 3 disjoint. Our options for groups of 4 begin PN, PM, PL, ON, OM, QK, QM, QN, QP. This isn't enough.

4) Columns 1 and 2, 3 and 4 disjoint. Our options for groups of 4 begin NL, NM, OK, OL, OM, PK, PL, PM, QL, QM, QO, QP, and we need to be able to complete all 12 of these. O occurs 4 times in the list above, and so needs to be paired with every letter possible in the sets ACDE} and {FHIJ}. However, O is paired with K, L, M, and in each of those A and F are blocked. Therefore, when O is paired with Q it must take both A and F, which is impossible (A and F have already been grouped together). Fails.

5) Three overlap in a single row, no disjointness - we know this works, since this is what comes from 74 if we take away a point from one of the groups of 5.

6). 3 overlap, one disjoint from one in the group of three. Our options for groups of 4 begin: QM, QL, QJ, QO, PM, PL, PJ, PO, OM, OL, OK, NL, NM. We cannot keep all 4 groups with P and all 4 groups with Q. PL, PM, and PJ and QL, QM, QJ must be paired with {FHI} and {CDE}, leaving only {A} unmatched for PO and QO. However, we cannot get rid of a P and a Q without getting rid of 2 potential groups of 4, leaving us with only 11. Fails.

7) 3 overlap, other two disjoint. Our options for groups of 4 begin: PM, PL, PJ, QM, QL, QJ, OK, OL, OM, NK, NL, NM, PO, PN, QO, QN. We can't keep all of these going simultaneously - we would only have room for 3 Ns, 3Os, 3Ps and 3Qs. This can only be accomplished by getting rid of the following 4 pairs: PO, PN, QO, QN. This still leaves 12 potential groups of 4, but each one must have a letter from

Case 1:				
A	A			
B		B		
C			C	
D				D
E				
	F	F		
	G		G	
	H			H
	I			
		J	J	
		K		K
		L		
			M	M
			N	
				O

Case 2:				
A		A		
B			B	
C				C
D				
E				
	F	F		
	G		G	
	H			H
	I			
	J			
		K	K	
		L		L
		M		
			N	N
			O	
				P

Case 3:				
A		A		
B			B	
C				C
D				
E				
	F		F	
	G			G
	H			
	I			
	J			
		K	K	
		L		L
		M		
		N		
			O	O
			P	
				Q

Case 4				
A		A		
B			B	
C				C
D				
E				
	F	F		
	G		G	
	H			H
	I			
	J			
		K		K
		L		
		M		
			N	N
			O	
			P	
				Q

Case 5:				
A	A	A		
B			B	
C				C
D				
E				
	F		F	
	G			G
	H			
	I			
		J	J	
		K		K
		L		
		M		
			N	N
			O	
				P

Case 6:				
A	A	A		
B			B	
C				
D				
E				
	F		F	
	G			G
	H			
	I			
		J	J	
		K		K
		L		
		M		
			N	N
			O	
				P
				Q

Case 7:				
A	A	A		
B			B	
C				C
D				
E				
	F		F	
	G			G
	H			
	I			
		J	J	
		K		K
		L		
		M		
			N	
			O	
				P
				Q

Case 8				
A	A	A		
		B	B	B
C			C	
D				D
E				
F				
	G		G	
	H			H
	I			
	J			
		K	K	
		L		L
		M		
			N	
			O	
				P
				Q

Table 2: Cases for 73 as 5 5's, 12 4's

{GHIJ} and {CDEF}. G cannot be paired with N or O, however, so it can really only be used twice (once for P and once for Q). Similarly, H cannot be paired with P or Q, and can only be used twice. I and J can each only be used 3 times (once for each of K, L and M), which leaves us short two. Fails.

8). Two groups of three share rows, the groups (must) overlap. Our options for groups of 4 begin: PM, PK, QM, QK, NL, NM, OL, OM, PO, PN, QO, QN. To continue with all 12 of these, they each need a letter from {GHIJ} and {CDEF}, respectively. But there are 4 Ps, and P can only be paired with three letters from each of those groups, so this fails.

So the columns of 5 must be in the right position for 73 in this form to be a subset of 74, and we must be in case 5 from above. If one of the groups of 4 can be extended to a group of 5 then there will be 6 5s and 11 4s, and we must have 74 in one of the only forms we can. This means that that group of 5 will contain N (the only option for a second cluster of three columns that will be disjoint from the first cluster - a necessary condition for 74).

So we assume that no group of 4 can be extended by adding N, and that no group of 4 containing N can be extended to a group of 5. We already have the groups N-BFJO and N-CGKP. The only other groups N could be in are of the form N- {LM} -{HI} - {DE} - Q. N must be in at least one other group of 4, so our options are the following: both NMIE and NLHD, both NMIE and NLHQ, just NMIE, and just NMIQ.

Our options for other groups of 4 begin: PL, PM, PJ, OL, OM, OK, and PO (without Q), and QL, QM, We can't complete all the ones with P and O - we must either drop PO, or one P and one O. If we drop an O and a P separately, we will get 5 without Q, and to get all 12 we will need all 5 with Q (note that this isn't even an option if N isn't in 2 groups of 4, or if one of those groups of 4 contains Q). To get all 5 with Q, however, we will need a QP and a QO - QA is impossible (unless we have QAOP, but we have retained OP above so this is impossible), and QN is impossible, so we need to be able to pair Q with all the other letters. But we run into the same problem - we cannot complete QO and QP with enough letters from { FGHI } or {BCDE} to finish (we have used them up in groups without Q), so this fails. Therefore, it never works to drop O and P separately - we must drop OP. This forces us to also take the group QOPA - otherwise we would have too few groups of 4.

Case 1 - we have both NMIE and NLHD.

Options for completing PL, PM, PJ, OL, OM, OK, i.e. the groups without Q:

PL - FE, IB ; PM - FD, HB; PJ - HE, ID ;

OL - GE, IB; OM - GD, HC; OK - HE, ID;

(arbitrarily, since O and P equivalent) becomes:

PLIB; PMFD; PJHE;

OLGE; OMHC; OKID;

QOPA;

QLF - C; QMG - B; QKH-B; QJI-C;

We cannot complete all the groups with Q, so this fails!

Case 2 - we have NMIE and NLHQ,

Our options for groups of 4 without Q, i.e. {PM, PL, PJ, OM, OL, OK}:

PM - FD, HB, HD; PL - FD, FE, IB, ID; PJ - HD, HE, ID;

OM - GD, HC, HD; OL - GD, GE, IC, ID; OK - HD, HE, ID;

options: 1)PM - FD; PL - IB; PJ - HE; OM - HC; OL - GE; OK - ID;

or:

2)PM - HB; PL - FE ; PJ - ID; OM - GD; OL - IC; OK - HE;

Note: PM - HD is impossible.

Now in neither of these options do we have DL, but if we don't have DL we can extend D to the group NLHQ, so DL must appear together, and since nothing else is left, it must be with Q. But Q has already been with L, so this is impossible.

Case 3: NMIE, but N is not in another group of 4:

Our options for (other) groups of 4 without Q are:

PM - FD, HB, HD; PL - FD, FE, IB, ID, HB, HD, HE; PJ - HD, HE, ID;

OM - GD, HC, HD; OL - GD, GE, IC, ID, HC, HD, HE; OK - HD, HE, ID;

Options: 1)PM-FD; PL-IB; PJ-HE; OM-HC; OL-GE; OK -ID;

2) PM-HB; PL-FE; PJ-ID; OM-GD; OL-IC; OK-HE;

note - PM-HD is impossible.

We have AOPQ, and so we need to create 4 more groups of 4 containing Q. These will all have to be of the form Q-{JKLM}-{FGHI}-{BCDE}

Following option 1 above we get: QL-FC, FD, HD; QM-GB, GD; QK-FB, FE, HB; QJ-GD, HD, IC;

which must become: QL - HD; QM - GB; QK - FE; QJ-IC;

Following option 2 above, we get: QL-GB, HD; QM - FC, FD, GC; QK - FD, IB, ID; QJ - GE, FC;

which must become: QL -HD; QM - FC; QK - IB; QJ - GE;

Both cases contain QLHD, which can be extended to a group of 5 by adding N, making it a subset of 74. We assumed that this is not possible, so this fails.

Case 4: just NIMQ. We can get as many as 6 more groups of 4 without Q, but Q can only be in 5 groups of 4 total, so we could only ever hope for 11 groups of 4 this way. Fails.

Therefore, every legal coloring contains a 74, or a 73 that could be extended up to a 74. But since no legal coloring could be made illegal by extending the 73 in this way, we may assume that every legal coloring contains a 74.

4 Appendices

4.1 Case 4, section 1.1

What if we allow disjoint columns - could we maybe achieve 6,5,5,5? We could only have at most 2 disjoint columns, so our options are as follows (columns numbered 1,2,3,4 as above, i.e. column 1 has length 6, columns 2,3,4 have length 5).

subcase 1) 1 disjoint from 2, everyone else connected - uses 16 of the rows. Q can only be in 5 columns, and there are at most 4 columns of 4 that can come from the rest, so we could only hope for 9 columns of size 4 - fails.

subcase 2) 2 disjoint from 3, everyone else connected - uses 16 of the rows. There are at most 6 we could hope for without Q (cannot be extended to 5s). The options for Q are:

Options for Q	total max points used	
5 columns of 4	$11 \cdot 4 + 2 \cdot 3$	$= 50$
2 columns of 5, 2 columns of 4	$2 \cdot 5 + 8 \cdot 4 + 3 \cdot 3$	$= 51$
3 columns of 5	$3 \cdot 5 + 6 \cdot 4 + 4 \cdot 3$	$= 51$

We cannot make 52 more points, so we can't achieve 73. FAIL.

Uses all 17 rows:

subcase 3) 1 disjoint from 2, and 3 disjoint from 4 - only 9 we could hope for - FAIL

subcase 4) 1 disjoint from 2, and 2 disjoint from 3 - only 6 we could hope for - FAIL

subcase 5) 2 disjoint from 3, and 3 disjoint from 4 - only 9 we could hope for - FAIL

subcase 6) 1, 2,3 overlap in a single row, 4 disjoint from 1 - only 9 we could hope for - FAIL

subcase 7) 1,2,3 overlap in a single row, 4 disjoint from 2 - only 9 we could hope for - FAIL

subcase 8) 2,3,4 overlap in a single row, 1 disjoint from 2 - only 9 we could hope for - FAIL

subcase 1			
A		A	
B			B
C			
D			
E			
F			
	G	G	
	H		H
	I		
	J		
	K		
		L	L
		M	
		N	
			O
			P

subcase 2			
A	A		
B		B	
C			C
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	K
		L	
		M	
		N	
			O
			P

subcase 3			
A		A	
B			B
C			
D			
E			
F			
	G	G	
	H		H
	I		
	J		
	K		
		L	
		M	
		N	
			O
			P
			Q

subcase 4			
A		A	
B			B
C			
D			
E			
F			
	G		G
	H		
	I		
	J		
	K		
		L	L
		M	
		N	
		O	
			P
			Q

subcase 5			
A	A		
B		B	
C			C
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	
		L	
		M	
		N	
			O
			P
			Q

subcase 6			
A	A	A	
B			
C			
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	K
		L	
		M	
		N	
			O
			P
			Q

subcase 7			
A	A	A	
B			B
C			
D			
E			
F			
	G		
	H		
	I		
	J		
		K	K
		L	
		M	
		N	
			O
			P
			Q

subcase 8			
A		A	
B			B
C			
D			
E			
F			
	G	G	G
	H		
	I		
	J		
	K		
		L	
		M	
		N	
			O
			P
			Q

Table 3: Illustrations for the subcases of case 4, first appendix

4.2 Cases 1 and 3 and others, from constructing 74

4.2.1 Case 1

One group of 3 columns overlap in a single row, all other columns overlap in distinct rows. We need 11 groups of 4, which we almost have because Q can be in as many as 5 groups of 4, and we have the openings PM, PJ, OM, OK, NM, NL.

A	A	A			
B			B		
C				C	
D					D
E					
	F		F		
	G			G	
	H				H
	I				
		J	J		
		K		K	
		L			L
		M			
			N	N	
			O		O
				P	P

It turns out it's not possible to simultaneously extend all of these pairs to full groups of 4. PJ, OK and NL all have to be paired with I. This lets us extend all of these to groups of three: PMF, PJI, OMG, OKI, NMH, NLI. But when we try to assign a 4th letter, we notice that A cannot be paired with any of these. B cannot be added to anything with F, J, N or O, so it's out. Similarly, C and D are impossible. So we can't get more than three of these, and so we can't complete 74 in this way.

4.2.2 Case 3

Two overlapping groups of 3 columns overlap in a single row, all other columns overlap in distinct rows.

A	A	A			
		B	B	B	
C			C		
D				D	
E					E
F					
	G		G		
	H			H	
	I				I
	J				
		K			K
		L			
		M			
			N		N
			O		
				P	P
				Q	

Working from the bottom as usual, our possibilities for groups of 4 are PM, PL, NM, NL, OM, OK, OL, QM, QL, QK, or something involving two or more of the letters {NOPQ}. We only have 10 of these, so we will need at least one group of 4 from this second category. Our options are QO, QN and PO, and we can't use these to take 3 from this group. Note that A and B are done - they can't be involved in any more groups of 4. So when we add 3rd and 4th letters to these pairs they need to be from the sets {CDEF} and {GHIJ}. O cannot be matched with C or G, P cannot be matched with H, I or D,E, Q cannot be matched with H or D and N cannot be matched with G, I and C, E, so we need to remove some of the pairs. Each pair with a letter in the first category uses two of these letters, and in the second category takes one. Even if we keep all the ones in the second category, we will end up using more than the number of other letters we have, and so we fail.

If we are only looking for 10 groups of 4 and 1 group of 3, we can achieve that by taking all 10 of the pairs in the first category. Note that, in fact, we can't achieve that by including any pairs in the second category. We will have to drop one in the first category for each letter we add. For example, if we include QO, we have to drop a Q and an O, although if we then take QN we don't have to drop another Q. Still, there's no way to even break even from this kind of deal. So the only way of achieving 10 groups of 4 is if we complete all 10 of these pairs from the first category: PM, PL, NM, NL, OM, OK, OL, QM, QL, QK. Look at just the first 4 options: PM-GJ-CF; PL-GJ-CF; NM-HJ-DF; NL-HJ-DF.

Up to the symmetry of M and L, this must be completed as follows: PMGF; PLJC; NMJD; NLHF.

We then need to add OM-HI-DE; OL-I-DE; QM-I-CE; QL-GI-CE, and this forces OKJF; QKJF, but it's impossible to have both. Therefore, we cannot even complete 10 groups of 4, and so this fails.

4.2.3 Cases 7-9

Case 7

A		A			
B			B		
C				C	
D					D
E					
	F	F			
	G		G		
	H			H	
	I				I
	J				
		K	K		
		L		L	
		M			M
			N	N	
			O		O
				P	P

Case 8

A		A			
B			B		
C				C	
D					D
E					
	F	F			
	G		G		
	H			H	
	I				I
	J				
		K		K	
		L			L
		M			
			N	N	
			O		O
			P		
				Q	Q

Case 9

A			A		
B				B	
C					C
D					
E					
	F	F			
	G		G		
	H			H	
	I				I
	J				
		K	K		
		L		L	
		M			M
		N			
			O	O	
			P		P
				Q	Q

Case 7): Q can be in at most 5 groups of 4. Then we need at least 6 more groups of 4 without Q. They can only be from the groups {NOP}, {KLM}, {GHIJ} and {BCDE}. But P can only be with K, O can only be with L, and N can only be with M. So there can only be 3 groups like this, and so this fails. Note - this holds even if we are only looking for 10 groups of 4 and a group of 3.

Case 8): Q can only be in two (additional) groups of 4, and the rest of the groups of 4 must be from {NOP}, {KLM}, {GHIJ} and {BCDE}. N and O can only be in two such groups, and P can only in 3, for a total of 7. This still only gets us to 9, and so we fail. Note - this holds even if we are only looking for 10 groups of 4 and 1 group of 3.

Case 9): Any group of 4 must come from the groups {OPQ}, {KLMN}, {GHIJ} and {BCDE}. O, P and Q can each only be in 2 groups, so we only get 6 potential groups and we fail. Note - this holds even if we are only looking for 10 groups of 4 and 1 group of 3.

4.3 Achieving 68 and 69 with a 7 in one column

If 68 has a 7, that leaves 10 unused rows from which we want to take groups of 2, 3 and 4, corresponding to columns of height 3, 4 and 5 - we are assuming we won't have a problem adding a last element from the group of 7 to each of these columns. For the case $i = 0$, we need 13 groups of 3 and 3 groups of 2, and we can do this (not unique):

A	A	A	A									
				B	B	B						
C							C	C	C			
D										D	D	D
	E			E			E			E		
	F				F			F			F	
		G		G					G			G
		H				H	H				H	
			I		I				I	I		
			J			J		J				J

Extra pairs of 2: BC, BD, EJ, FG, HI.

When i=1, we want one group of 4, 11 groups of 3 and 4 groups of 2 for a total of 16 columns. We can do this by modifying the group above - if we add B to the set {A, C, D} then we gain a group of 4, without sacrificing any of the groups of three, and so in fact we do even better than we need - we can achieve 69 in this way.

When i=2, we want two groups of 4, 9 groups of 3 and 5 groups of 2. We can do this if the two groups of 4 overlap:

A	A	A								
B			B	B						
C					C	C	C			
D								D	D	D
	E				E			E		
	F		F			F			F	
	G						G			G
		H		H	H				H	
		I	I				I	I		
				J		J				J

Extra groups of 2: AJ, BE, BG, EJ, GH, IJ.

When i=3, we want 3 groups of 4, 7 groups of 4 and 6 groups of 2.

A	A	A							
B			B						
C				C	C	C			
D							D	D	D
	E			E			E		
	F		F		F			F	
	G					G			G
		H		H				H	
		I	I			I	I		
		J			J				J

Extra groups of 2: BE, BG, BH, BJ, EJ, GH.

For 69, we've already seen that we can achieve it when i=1. When i=0, we have a simple problem - every row can only be in (at most) 4 groups of 3, and so for every row, there is one row it cannot be paired with. Since we have ten rows, that means that there are 5 pairs (at least) that will be missing. But we only have $\binom{10}{2} = 45$ pairs total, and to form 14 groups of three we will need $14 \cdot 3 = 42$ pairs. If there are 5 pairs missing, this becomes impossible.

4.4 72 has less than 6 in any row, column

Our options for the largest columns are:

Sum to 27:	Sum to 26:	Sum to 25	Sum to 24:
6 6 6 5 4	6 6 6 4 4	6 6 5 4 4	6 6 4 4 4
6 6 5 5 5	6 6 5 5 4	6 5 5 5 4	6 5 5 4 4
	6 5 5 5 5	5 5 5 5 5	5 5 5 5 4

If the first two columns are both 6, this leaves either 5 or 6 unused letters, from which we need to take pairs to complete groups of 4 (as standard, we will show that $6, 6, \underbrace{4\dots4}_{15}$ doesn't work, and then argue by convexity that $6, 6, \underbrace{5\dots5}_{i}, \underbrace{4\dots4}_{15-2i}, \underbrace{3\dots3}_{i}$ cannot work either for all values of i). If there are only 5 unused letters there are only 10 pairs, and so we cannot possibly hope to get 15 groups of 4. If there are 6 unused letters there are 15 potential pairs, but it turns out we cannot complete them all to groups of 4.

If there are 6 unused letters, the groups of 6 overlap in the following way: {A,B,C,D,E,F} and {A,G,H,I,J,K}, and we will call the unused letters L, M, N, O, P and Q. We need all 15 pairs to be filled out. Once we fill the first five (arbitrarily, we could have listed all the ones with M first, etc.

L M	L N	L O	L P	L Q	M N	M O	M P	M Q	N O	N P	N Q	O P	O Q	P Q
B	C	D	E	F					B1	B2	B3	B3	B2	B1
						C1	C2	C3				C3	C2	C1
					D1		D2	D3		D3	D2			D1
					E1	E2		E3	E3		E2		E1	
					F1	F2	F3		F3	F2		F1		

We need to find two completely disjoint ways of filling out the 5 different letters (disjoint because we need to also put the letters {G,H,I,J,K} in, and they can't be paired with their corresponding letters {B,C,D,E,F}). There are 6 possibilities: B1, C2, D3, E2, F1; B1,C3, D2, E1, F2; B2, C1, D2, E3, F1; B2, C3, D1, E2, F3; B3, C1, D3, E1, F3; and B3, C2, D1, E3 F2. No two of these are disjoint, and so there is no way to complete all 15 groups of 4. If we have groups of 3 and 5, in addition to groups of 4, we use more than 15 pairs and so the opening 6,6 is impossible with 72.

The other possibility is that the first three columns are 6,5,5. If this is the case, we need 14 more groups of 4. There are 41 cases for the first 4 columns 6554, broken down into a few main categories: 655 all share a single row, 655 all overlap distinctly; 6-5 disjoint; 5-5 disjoint; 6 disjoint from both 5s (but they overlap); 6-5 disjoint and 5-5 disjoint. The final case of these is that all three are disjoint from each other - this takes up 16 of the 17 rows, and so any group of 4 would have to use the remaining row, Q. There can only be 5 of these, so this last one clearly fails and we don't need to take it into consideration.

We use the principle that, since there is a group of 6, if row Q is unused then it can only be in at most 5 groups of 4, and if both P and Q are unused they can only be in as many as 9 groups of 4 together. Then to get a bound on the number of groups of 4 made up entirely of used rows, for each column look at the number of elements that only appear in that column - those are the only ones we can use in a group of 4 made up entirely of used elements. We multiply the smallest of these two numbers to get a bound on the number of these such groups. If the sum of these bounds is less than 13 (we need 14 groups of 4, but one is already listed), then we have no hope of achieving 72 in this way. Cases marked with a ✠ fail because of this principle - cases not marked need special treatment, which they will receive after the list of all 41 cases.

655 overlap in a single row:

✠ Case 1:			
A	A	A	
B			B
C			
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	K
		L	
		M	
		N	
			O

✠ Case 2:			
A	A	A	
B			B
C			
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	
		L	
		M	
		N	
			O
			P

✠ Case 3:			
A	A	A	
B			
C			
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	K
		L	
		M	
		N	
			O
			P

✠ Case 4:			
A	A	A	
B			B
C			
D			
E			
F			
	G		
	H		
	I		
	J		
		K	
		L	
		M	
		N	
			O
			P
			Q

✠ Case 5:			
A	A	A	A
B			
C			
D			
E			
F			
	G		
	H		
	I		
	J		
		K	
		L	
		M	
		N	
			O
			P
			Q

✠ Case 6:			
A	A	A	
B			
C			
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	
		L	
		M	
		N	
			O
			P
			Q

655 overlap distinctly:

Case 7:				Case 8:				Case 9:				Case 10:			
A	A			A	A			A	A			A	A		A
B		B		B		B		B		B		B		B	
C			C	C			C	C				C			
D				D				D				D			
E				E				E				E			
F				F				F				F			
	G	G			G	G			G	G			G	G	
	H		H		H		H		H		H		H		
	I				I				I				I		
	J				J				J				J		
		K	K			K				K	K			K	
		L				L				L				L	
		M				M				M				M	
			N				N				N				N
							O				O				O
															P

Case 11:				Case 12:				✠ Case 13:				✠ Case 14:			
A	A			A	A			A	A			A	A		
B		B		B		B		B		B		B		B	
C				C			C	C				C			
D				D				D				D			
E				E				E				E			
F				F				F				F			
	G	G	G		G	G			G	G			G	G	
	H				H				H		H		H		
	I				I				I				I		
	J				J				J				J		
		K				K				K				K	
		L				L				L				L	
		M				M				M				M	
			N				N				N				N
			O				O				O				O
			P				P				P				P
															Q

6-5 disjoint:

✠ Case 15:				✠ Case 16:				✠ Case 17:				✠ Case 18:			
A		A		A		A		A		A		A		A	
B			B	B			B	B			B	B			
C				C				C				C			
D				D				D				D			
E				E				E				E			
F				F				F				F			
	G	G			G	G			G	G			G	G	
	H		H		H		H		H				H		H
	I				I				I				I		
	J				J				J				J		
	K				K				K				K		
		L	L			L				L	L			L	L
		M				M				M				M	
		N				N				N				N	
			O				O				O				O
							P				P				P

✠ Case 19:				✠ Case 20:				✠ Case 21:				✠ Case 22:			
A		A	A	A		A	A	A		A		A		A	
B				B				B			B	B			
C				C				C				C			
D				D				D				D			
E				E				E				E			
F				F				F				F			
	G	G			G	G			G	G	G		G	G	G
	H		H		H				H				H		
	I				I				I				I		
	J				J				J				J		
	K				K				K				K		
		L				L				L				L	
		M				M				M				M	
		N				N				N				N	
			O				O				O				O
			P				P				P				P
							Q								Q

6-5 disjoint, continued:

✠ Case 23:				✠ Case 24:				✠ Case 25:			
A		A		A		A		A		A	
B			B	B				B			
C				C				C			
D				D				D			
E				E				E			
F				F				F			
	G	G			G	G			G	G	
	H				H		H		H		
	I				I				I		
	J				J				J		
	K				K				K		
		L				L				L	L
		M				M				M	
		N				N				N	
			O				O				O
			P				P				P
			Q				Q				Q

5-5 disjoint:

✠ Case 26:			
A	A		
B		B	
C			C
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	K
		L	
		M	
		N	
			O

✠ Case 27:			
A	A		
B		B	
C			C
D			
E			
F			
	G		
	H		
	I		
	J		
		K	K
		L	
		M	
		N	
			O
			P

✠ Case 28:			
A	A		
B		B	
C			
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	K
		L	
		M	
		N	
			O
			P

✠ Case 29:			
A	A		A
B		B	
C			
D			
E			
F			
	G		
	H		
	I		
	J		
		K	K
		L	
		M	
		N	
			O
			P

✠ Case 30:			
A	A		A
B		B	
C			
D			
E			
F			
	G		
	H		
	I		
	J		
		K	
		L	
		M	
		N	
			O
			P
			Q

✠ Case 31:			
A	A		
B		B	
C			C
D			
E			
F			
	G		
	H		
	I		
	J		
		K	
		L	
		M	
		N	
			O
			P
			Q

✠ Case 32:			
A	A		
B		B	
C			
D			
E			
F			
	G		G
	H		
	I		
	J		
		K	
		L	
		M	
		N	
			O
			P
			Q

6-5-5 disjoint:

✠ Case 33:				✠ Case 34:				✠ Case 35:			
A		A		A		A		A		A	
B			B	B			B	B			B
C				C				C			
D				D				D			
E				E				E			
F				F				F			
	G		G		G		G		G		
	H				H				H		
	I				I				I		
	J				J				J		
	K				K				K		
		L	L			L				L	L
		M				M				M	
		N				N				N	
		O				O				O	
			P				P				P
							Q				Q

✠ Case 36:				✠ Case 37:			
A		A		A		A	A
B				B			
C				C			
D				D			
E				E			
F				F			
	G		G		G		G
	H				H		
	I				I		
	J				J		
	K				K		
		L	L			L	
		M				M	
		N				N	
		O				O	
			P				P
			Q				Q

5-6-5 disjoint:

✠ Case 38:				✠ Case 39:				✠ Case 40:				✠ Case 41:			
A			A	A			A	A				A			A
B				B				B				B			
C				C				C				C			
D				D				D				D			
E				E				E				E			
F				F				F				F			
	G	G			G	G			G	G			G	G	G
	H		H		H		H		H		H		H		
	I				I				I				I		
	J				J				J				J		
	K				K				K				K		
		L	L			L				L	L			L	
		M				M				M				M	
		N				N				N				N	
		O				O				O				O	
			P				P				P				P
							Q				Q				Q

The remaining cases are just 7-12. For cases 10, 11, and 12, we run into a problem trying to combine all of the ones with Q with all of the ones without Q. No matter how we create a group of 4 containing Q, it must contain some pair from the sets {HIJ }, {KLM }, and {NOP}, but we have already used all of those pairs, so every group with Q takes away one of the original nine.

In cases 8 and 9, we need all 9 with P and Q, as well as all 4 from {NO} and {IJ}. In case 9, to create the groups with {NO} and {IJ}, we must re-use the letters from the set {LM}, and so we cannot possibly create all possible groups of 4 with P and Q - there aren't enough groups of three available that don't use those pairs. In case 8, to create the groups with {NO} and {IJ}, we must re-use one of the letters from the set {KLM}, assume it is K. K then gets paired with N,O,I and J, so our only hope of using it in a group of 4 with P or Q is to use one of N,O,I,J with P and Q at the same time - for example, P,Q,N. Otherwise, there aren't enough letters to make PN, PO, PI, PJ and QN, QO,QI and QJ, because we can't add K to any of those pairs. This means, though, that we need to have PK and QK as two separate groups of 4, which doesn't work - K can only be paired with H and elements from the set { A,C,D,E,F }.

For case 7, we have that each of O, P and Q can each only be in at most 4 groups of 4 on their own. This in theory would be enough, though, if we could get also get both groups of 4 with N (and not O, P or Q). These groups would all be of the form {ABCDEF}-{GHIJ}-{KLMN}-{OPQ}. But we run into trouble trying to realize all 12 of these groups - G can only be paired with N, and so two of {OPQ} cannot have a group of 4 with G. We can try and get around this, though, by having OGN? and then PQG?. This reduces the number of groups from 14 to 13, which is just enough. In addition, however, K can only be paired with I and J, so one of {OPQ} cannot have a group of 4 containing K. This also reduces the number of groups of 4, and since we can't combine it with the reduction above we drop to 12 groups of 4, which is too small.

Therefore, a color class with 72 points cannot have 6 in any row or column.