Theorem 0.0.1 *For all a, $\sum_{i=0}^{k}(-1)^i\binom{k}{i}(a+i)^k = (-1)^k k!$*

Proof:

Let $f_j(x) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(a+i)^j x^{a+i}$.
We are interested in $f_k(1)$.
Note that the derivative of $f_j(x)$ is

$$f_j'(x) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(a+i)^j(a+i)x^{a+i-1} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(a+i)^{j+1}x^{a+i-1} =$$

$$\frac{1}{x}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(a+i)^{j+1}x^{a+i} = \frac{1}{x}f_{j+1}(x).$$

So

$$f_{j+1}(x) = xf_j'(x).$$

Claim: For all $j \in \mathbb{N}$ there exists a function $p_j(x)$ such that

- $f_j(x) = (-1)^j\frac{k!}{(k-j)!}x^{a+j}(1-x)^{k-j} + (1-x)^{k-j+1}p_j(x)$
- $p_j(x)$ is differentiable.
- $p_j(1)$ is defined (counterexample: $p_j(x)$ cannot be $1/(1-x)$.)

This is an easy proof by induction using that $f_{j+1}(x) = xf_j'(x)$.
Assuming the claim we proceed.
Note that there exists p_k such that $p_k(1)$ is defined and

$$f_k(x) = (-1)^k k! x^{a+k} + (1-x)p_k(x)$$

Recall that our sum is $f_k(1)$ which is easily seen to be $(-1)^k k!$. We needed that $p_k(1)$ was defined to make that last term be 0. ∎