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Banach-Mazur Games on Graphs

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ABSTRACT. We survey determinacy, definability, and complexity issues of Banach-Mazur games on finite and infinite graphs.
Infinite games where two players take turns to move a token through a directed graph, thus tracing out an infinite path, have numerous applications in different branches of mathematics and computer science. In the usual format, the possible moves of the players are given by the edges of the graph; in each move a player takes the token from its current position along an edge to a next position. In Banach-Mazur games the players instead select in each move a *path* of arbitrary finite length rather than just an edge. In both cases the outcome of a play is an infinite path. A winning condition is thus given by a set of infinite paths which is often specified by a logical formula, for instance from S1S, LTL, or first-order logic.
Banach-Mazur games have a long tradition in descriptive set theory and topology, and they have recently been shown to have interesting applications also in computer science, for instance for planning in nondeterministic domains, for the study of fairness in concurrent systems, and for the semantics of timed automata.
It turns out that Banach-Mazur games behave quite differently than the usual graph games. Often they admit simpler winning strategies and more efficient algorithmic solutions. For instance, Banach-Mazur games with ω-regular winning conditions always have positional winning strategies, and winning positions for finite Banach-Mazur games with Muller winning condition are computable in polynomial time.

1 Banach-Mazur Games

Game playing is a powerful metaphor that fits situations in which interaction between autonomous agents plays a central role. Indeed, numerous problems in computer science and other fields can be understood, mathematically treated, and solved in terms of appropriate mathematical models of games. There is of course a large variety of game models, leading to vastly different mathematical theories of games.

A prominent class of games, which is particularly useful for problems such as the synthesis and verification of interactive systems (with non-terminating behaviour and ongoing interaction between system and environment), or for the evaluation of fixed point logics and other important specification formalisms, are infinite games, where two players take turns to move a token through a directed graph thus tracing out an infinite path. The objectives of the players are given by suitable properties of infinite paths, often specified by logical formulae, for instance from monadic second order logic (S1S), linear-time temporal logic (LTL), or first-order logic (FO). Some central mathematical questions concerning such games are: Which games are determined (in the sense that from each position, exactly one player has a winning strategy)? How to compute winning positions? Are there optimal strategies, and if so, what is their complexity and how to compute them efficiently? How much knowledge of the play history is necessary to compute an optimal next action? In what logical formalisms can we define winning positions and winning strategies? And so on.

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These questions are not just of theoretical interest. They are in fact standard design and verification problems (of interactive systems) in purified form. For background on such methodologies, based on the interplay between logic, automata, and games, see e.g. [8].

In the usual format of infinite games on graphs, the possible moves of the players are given by the edges of the graph; in each move a player takes the token from its current position along an edge to a next position. Here we study a different variant of graph games where, in each move, the players select a path of arbitrary finite length rather than just an edge. We call these games Banach-Mazur games on graphs.

DEFINITION 1. A Banach-Mazur game $\mathrm{BM}(G, v, \mathrm{Win})$ is given by an a directed graph $G = (V, E)$ without terminal nodes, an initial position $v \in V$, and a winning condition $\mathrm{Win} \subseteq \mathrm{Paths}(G, v)$ where $\mathrm{Paths}(G, v) \subseteq V^\omega$ denotes the set of infinite paths through G that start at node v.

The game $\mathrm{BM}(G, v, \mathrm{Win})$ is played by two players, called Player 0 and Player 1. In the opening move, Player 0 selects a finite, non-empty path x_0 from v through G. The players take turns, extending in each move the finite path $x_0 x_1 \dots x_{m-1}$ played so far by a new segment x_m (which again has to be a non-empty and finite path). In an infinite number of moves, the players thus trace out an infinite path $\pi \in \mathrm{Paths}(G, v)$. Player 0 wins the play, if $\pi \in \mathrm{Win}$, otherwise Player 1 wins.

In somewhat different forms, Banach-Mazur games have been extensively studied in descriptive set theory (see [13, Chapter 6] or [14, Chapter 8.H]) and topology (see e.g. [21]). In their original variant (see [15, pp. 113–117]), the winning condition is a set W of real numbers; in the first move, one of the players selects an interval d_1 on the real line, then her opponent chooses an interval $d_2 \subset d_1$, then the first player selects a further refinement $d_3 \subset d_2$ and so on. The first player wins if the intersection $\bigcap_{n \in \omega} d_n$ of all intervals contains a point of W, otherwise her opponent wins.

A similar game can be played on any topological space. Let $\mathcal{V}$ be a family of subsets of a topological space X such that each $V \in \mathcal{V}$ contains a non-empty open subset of X, and each nonempty open subset of X contains an element $V \in \mathcal{V}$. In the Banach-Mazur game defined on $X, \mathcal{V}$ with winning winning condition $W \subseteq X$, the players take turns to choose sets $V_0 \supset V_1 \supset V_2 \supset \dots$ in $\mathcal{V}$, and Player 0 wins the play if $\bigcap_{n<\omega} V_n \cap \mathrm{Win} \neq \emptyset$. We refer to [21] for a survey on topological games and their applications to set-theoretical topology. Notice that Banach-Mazur games on graphs are just a special case of this general topological setting. Indeed, the set $\mathrm{Paths}(G, v)$ of infinite paths through G from v is a topological space whose basic open sets are $\mathcal{O}(x)$, the sets of infinite prolongations of some finite path $x \in \mathrm{FinPaths}(G, v)$. Thus, when a player prolongs the finite path x played so far to a new path xy, she reduces the set of possible outcomes of the play from $\mathcal{O}(x)$ to $\mathcal{O}(xy)$, and she wins an infinite play $x_0 x_1 \dots$ if, and only if $\bigcap_{n<\omega} \mathcal{O}(x_0 \dots x_{n-1}) \cap \mathrm{Win} \neq \emptyset$.

Applications of Banach-Mazur games. Banach-Mazur games on graphs have recently appeared in several application areas in computer science. Pistore and Vardi used a variation of Banach-Mazur games for planning in nondeterministic domains [20]. In their scenario, the desired infinite behaviour of a system, which should be enforced by a plan, is specified by formulae in linear temporal logic LTL. It is assumed that the outcome of actions may be nondeterministic. Hence a plan does not have only one possible execution path in the planning domain, but an execution tree. Between weak planning (some possible execution path satisfies the specification) and strong planning (all possible outcomes are consistent with the specification), there is a spectrum of intermediate cases such as strong cyclic planning: every possible partial execution of the plan can be extended to an execution reaching the desired goal. In this context, planning can be modelled by a game between a friendly player E and a hostile player A selecting the outcomes of nondeterministic actions. The game is played on the execution tree of the plan, and the question is whether the friendly player E has a strategy to ensure that the outcome (a path through the execution tree) satisfies the given LTL-specification. In contrast to the general scenario of Banach-Mazur games, the main interest here are games with finitely many alternations between players. Pistore and Vardi show that the planning problems in this context can be solved by automata-based methods in 2EXPTIME.

Banach-Mazur games appear also in the characterisation of fair behaviour in concurrent systems. There are many different notions of fairness. A very convincing one [23] defines a fairness property in a transition system as a set of (infinite) runs that is topologically large (co-meager). This is equivalent to say that, in an associated Banach-Mazur game, the first player (the scheduler) has a winning strategy to ensure fairness. It is a consequence of the positional determinacy of Banach-Mazur games with ω-regular winning conditions (see Theorem 17 below) that, on finite graphs, ω-regular fairness properties coincide with ω-regular properties that are probabilistically large under positive Markov measures. Hence , any ω-regular fairness property has probability one under randomised scheduling. As a further consequence, one can use results about finite Markov chains for checking whether a finite system is fairly correct with respect to LTL or ω-regular specifications.

Finally, Banach-Mazur games have recently been used to describe the semantics of timed automata [1, 2]. Timed automata are an important model for verification, but for many purposes, its idealized mathematical features such as infinite precision, instantaneous events lead to violations of specifications due to unlikely sequences of events. Therefore alternative semantics for the satisfaction of LTL specifications have been proposed, based on probability or on topological largeness, to rule out unlikely runs. By means of Banach-Mazur games, it has been established, that the two semantics coincide.

Here we study Banach-Mazur games on graphs, and focus on the above-mentioned central mathematical questions, such as determinacy, the structure and algorithmic properties of winning strategies, and the definability of winning regions.

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2 Topology and determinacy

For any arena (G, v) of a Banach-Mazur game, the space $\text{Paths}(G, v)$ is endowed with a topology whose basic open sets are $\mathcal{O}(x)$, the sets of infinite prolongations of some finite path $x \in \text{FinPaths}(G, v)$. A set $X \subseteq \text{Paths}(G, v)$ is *open* if it is a union of basic open sets $\mathcal{O}(x)$, i.e., if $X = W \cdot V^\omega \cap \text{Paths}(G, v)$ for some set $W \subseteq V^*$. A tree $T \subseteq \text{FinPaths}(G, v)$ is a set of finite paths that is closed under prefixes. It is easily seen that $X \subseteq \text{Paths}(G, v)$ is *closed* (i.e., the complement of an open set) if, and only if, it is the set of infinite branches of some tree T, denoted $X = [T]$. Notice that $\text{Paths}(G, v)$ itself is a closed set in the space V^ω, the set of all infinite sequences on V.

The class of *Borel sets* is the closure of the open sets under countable union and complementation. Borel sets form a natural hierarchy of classes $\mathbf{\Sigma}^0_\eta$ for $1 \leq \eta < \omega_1$, whose first levels are $\mathbf{\Sigma}^0_1$ (or G), the collection of all open sets, $\mathbf{\Pi}^0_1$ (or F), the closed sets, $\mathbf{\Sigma}^0_2$ (or F_σ), the countable unions of closed sets, and $\mathbf{\Pi}^0_2$ (or G_δ), the countable intersections of open sets. In general, $\mathbf{\Pi}^0_\eta$ contains the complements of the $\mathbf{\Sigma}^0_\eta$-sets, $\mathbf{\Sigma}^0_{\eta+1}$ is the class of countable unions of $\mathbf{\Pi}^0_\eta$-sets, and $\mathbf{\Sigma}^0_\lambda = \bigcup_{\eta<\lambda} \mathbf{\Sigma}^0_\eta$ for limit ordinals λ.

We recall that a set X in a topological space is *dense*, if its intersection with every (basic) non-empty open set is non-empty.

LEMMA 2. *For any strategy g of Player 1 in a Banach-Mazur game on a graph (G, v), the set $\text{Plays}(g)$ of all plays that are consistent with g is a countable intersection of dense open sets.*

PROOF. Clearly, $\text{Plays}(g) = \bigcap_{n\in\omega} \text{Plays}_n(g)$ where $\text{Plays}_n(g)$ is the set of all plays that may arise if Player 1 moves according to g during her first n moves. Obviously, $\text{Plays}_n(g)$ is open. But it is also dense, since every finite path x can be used by Player 0 as her opening move, so there must be a prolongation of x in $\text{Plays}_n(g)$, which means that $\mathcal{O}(x) \cap \text{Plays}_n(g) \neq \emptyset$. ∎

Notice that, if $X \subseteq \text{Paths}(G, v)$ is a dense open set, then any finite path x has a finite prolongation xy such that $\mathcal{O}(xy) \subseteq X$. In a topological sense, the dense open sets are large sets, and so is any countable intersection of such. Hence, by any strategy in a Banach-Mazur game, Player 1 can exclude only a topologically small set of plays. This means that she can only have a winning strategy if the set Win of winning plays for Player 0 is small, and her own set of winning plays, $\text{Paths}(G, v) \setminus W$, is large.

For strategies of Player 0, the situtation is slightly different, since she starts the play. Hence, for any strategy f of Player 0, $\text{Plays}(f) \subseteq \mathcal{O}(x)$ where x is the opening move by f. After the first move, the remaining game is one where the role of the players have been switched (i.e. Player 1 now moves first). By the same argument as in the previous lemma we infer that the set of plays consistent with a strategy of Player 0 is large inside some basic open set of plays.

LEMMA 3. *For any strategy f of Player 0 in a Banach-Mazur game, $\text{Plays}(f)$ is a countable intersection of dense open subsets of $\mathcal{O}(x)$, where x is the opening move by f.*

The observations that we made on the set of plays that are consistent with strategies in Banach-Mazur games give a quite precise characterisation, in term of topological notions, of the games for which Players 0 and 1 have winning strategies.

A set in a topological space is *nowhere dense* if it is not dense in any open set or, equivalently, if its complement contains a dense open set. A set is *meager* (or topologicaly small) if it is a union of countably many nowhere dense sets, and *co-meager* (or topologically large) if its complement is meager. A topological space is called a *Baire space* if no non-empty set is both open and meager, or equivalently, if any countable intersection $X = \bigcap_{n<\omega} X_n$ of dense open sets X_n is dense. The spaces $\text{Paths}(G, v)$ are Baire spaces since, for any finite path x, we find an infinite extension $xy_0y_1 \cdots \in X$ by choosing, for each n, a finite prolongation $xy_0 \ldots y_n$ of $xy_0 \ldots y_{n-1}$ such that $\mathcal{O}(xy_0 \ldots y_n) \subseteq X_n$. In Baire spaces a set is co-meager if, and only if, it contains a dense $\mathbf{\Pi}^0_2$ set.

Hence we have shown that, in Banach-Mazur games, $\text{Plays}(g)$ is co-meager for every strategy g of Player 1 and $\text{Plays}(f)$ is co-meager in some basic open set for every strategy f of Player 0. Conversely, for any meager set $W \subseteq \text{Paths}(G, v)$, Player 1 has a strategy g such that $\text{Plays}(G) \cap W = \emptyset$. Indeed, if $W = \bigcup_{n<\omega} X_n$ with X_n nowhere dense, then in her n-th move, Player 1 prolongs the path constructed so far to a path x_n such that $\mathcal{O}(x_n) \cap X_n = \emptyset$ which is always possible since the complement of X_n contains a dense open set. Clearly every play consistent with this strategy avoids W. Analogously, for every set that is co-meager in some basic open set, Player 0 has a strategy f such that $\text{Plays}(f) \subseteq W$.

Our observations are summarized by the Banach-Mazur-Theorem which gives a precise characterisation of the games where Player 0 or Player 1 has a winning strategy.

THEOREM 4.*[Banach-Mazur]*

(1) Player 1 *has a winning strategy for the game* $\text{BM}(G, v, \text{Win})$ *if, and only if,* $\text{Win} \subseteq \text{Paths}(G, v)$ *is meager.*

(2) Player 0 *has a winning strategy for* $\text{BM}(G, v, \text{Win})$ *if, and only if, there exists a finite path* $x \in \text{FinPaths}(G, v)$ *such that* $\mathcal{O}(x) \setminus \text{Win}$ *is meager in* $\text{Paths}(G, v)$ *(i.e.,* Win *is co-meager in some basic open set).*

This result appears, in different terms, in the Scottish Book [15, Problem 43] where it is mentioned as a conjecture due to Mazur, with an addendum by Banach, dated August 4, 1935 saying that "Mazur's conjecture is true". The Banach-Mazur-Theorem was published for the first time by Mycielski, Świerczkowski, and Zieba [18], without proof; the first published proof is due to Oxtoby [19].

¿From Theorem 4 we easily get strong results on determinacy of Banach-Mazur games.

COROLLARY 5. *Every Banach-Mazur game* $\text{BM}(G, v, \text{Win})$ *such that* $\text{Win} \subseteq \text{Paths}(G, v)$ *has the Baire property is determined.*

Recall that a set X in a topological space has the *Baire property* if its symmetric difference with some open set is meager. Since Borel sets have the Baire property, it follows that Banach-Mazur games are determined for Borel winning conditions. Standard winning conditions used in computer science applications (in particular the ω-regular winning conditions) are contained in very low levels of the Borel hierarchy.

A converse to Corollary 5 in terms of specific games does not hold. Indeed one can construct determined games with winning conditions of arbitrary complexity by combining a trivial game won by Player 0 with an arbitrarily complex game in such a way that Player 0 can avoid the complicated part.

A more interesting question is whether one can prove a converse for winning conditions that *guarantee determinacy* in the following sense. Let $W \subseteq C^\omega$ be a set of infinite words on some alphabet C. On every graph $G = (V, E)$ equipped with a function $\Omega : V \to C$, the set W defines a winning condition $\Omega^{-1}(W) := \{\pi \in \text{Paths}(G, v) : \Omega(\pi) \in W\}$. We then say that W guarantees determinacy for Banach-Mazur games if *all* games with a winning condition $\Omega^{-1}(W)$ are determined.

We can link the Baire property with the determinacy of Banach-Mazur game in the following class-wise sense.

THEOREM 6. *For every class $\Gamma \subseteq \mathcal{P}(C^\omega)$ the following are equivalent.*
(1) All winning conditions $W \in \Gamma$ guarantee determinacy for Banach-Mazur.
(2) All sets $W \in \Gamma$ have the Baire property.

PROOF. If $W \subseteq C^\omega$ has the Baire property then so has $\Omega^{-1}(W)$, for all functions $\Omega : V \to C$ that label the nodes of a graph $G = (V, E)$ with elements of C. Thus, by Corollary 5 W guarantees determinacy.

For the converse, suppose that $W \subseteq C^\omega$ does not have the Baire property. To construct a non-determined game, let $G(C)$ be the complete directed graph on C itself (and let Ω be the identity function on C). We do not use W directly as a winning condition, but modify it as follows. Let $S := \{x \in C^* : \mathcal{O}(x) \setminus W \text{ is meager}\}$ and let Z be the symmetric difference of W with the open set $Y = \bigcup_{x \in S} \mathcal{O}(x)$.

We claim that the Banach-Mazur game on $G(C)$ with winning condition Z is not determined. Since Z is the symmetric difference of W with an open set, it cannot be meager (otherwise W would have the Baire property), hence Player 1 does not have a winning strategy. So suppose that Player 0 has a winning strategy. This can only happen if Z is co-meager in some basic open set $\mathcal{O}(x)$. For $x \in S$, this is impossible since $\mathcal{O}(x) \cap Z = \mathcal{O}(x) \setminus W$ is meager. Hence $x \in C^* \setminus S$. But then $\mathcal{O}(x) \cap Y = \emptyset$. Otherwise we would have some $y \in S$ such that $\mathcal{O}(x) \cap \mathcal{O}(y) \neq \emptyset$, which means that $\mathcal{O}(x) \subseteq \mathcal{O}(y)$ or $\mathcal{O}(y) \subseteq \mathcal{O}(x)$. In either case, since $\mathcal{O}(y) \cap Z = \mathcal{O}(y) \setminus W$ is meager, Z cannot be co-meager in $\mathcal{O}(x)$.

Now, since $\mathcal{O}(x) \cap Y = \emptyset$, we have $\mathcal{O}(x) \cap Z = \mathcal{O}(x) \cap W$, and if this set were co-meager in $\mathcal{O}(x)$ then $x \in S$, a contradiction.

Thus, none of the players has a winning strategy. ∎

A specific example of a non-determined Banach-Mazur game can be obtained by modifying a well-known construction on the basis of ultrafilters. Let G_2 be the complete directed graph with vertices 0,1, and for any set $U \subseteq \mathcal{P}(\omega)$, let let W_U be the set of infinite sequences $x_0 x_1 x_2 \cdots \in \{0, 1\}^\omega$ such that $\{n : x_n = 0\} \in U$.

An ultrafilter over ω is a set $U \subseteq \mathcal{P}(\omega)$ that does not contain $\emptyset$, that includes with any set also all its supersets, with any two sets also their intersection, and such that for any set $x \subseteq \omega$ either $x \in U$ or $\omega \setminus x \in U$. An ultrafilter is free if it contains all co-finite sets. As a consequence, it does not contain any finite set. The Boolean Prime Ideal Theorem (a weak form of the Axiom of Choice) implies that free ultrafilters exist.

PROPOSITION 7. *If U is a free ultrafilter, then the Banach-Mazur game on G_2 with winning condition W_U is not determined.*

PROOF. Without loss of generality, we may assume that Player σ plays in each move a finite word in σ^+. Hence the game is equivalent to the game where the players play a strictly increasing sequence $a_0 < a_1 < a_2 < \dots$ and Player 0 wins the resulting infinite play if, and only if, the set $[0, a_0) \cup [a_1, a_2) \cup [a_3, a_4) \cup \dots$ belongs to U.

Assume that Player 0 has a winning strategy f which maps any increasing sequence $a_0 < a_1 < \cdots < a_{2n-1}$ of even length to $a_{2n} = f(a_0 a_1 \dots a_{2n-1}) > a_{2n-1}$. We consider two intertwined counter-strategies of Player 1, essentially forcing Player 0 to simultaneously perform two plays against herself. In reply to the first move a_0, Player 1 selects an arbitrary $a_1 > a_0$ and then sets up the two plays as follows: In the first one she replies to a_0 by a_1 and waits for the answer $a_2 = f(a_0 a_1)$ by Player 0. She then uses a_2 as her own reply to a_0 in the second play and gets the answer $a_3 = f(a_0 a_2)$ by Player 0, which she now uses as her next move in the first play. There Player 0 responds by $a_4 = f(a_0 a_1 a_2 a_3)$ which is again used by Player 1 as her answer to $a_0 a_2 a_3$ in the second play. And so on.

In this way, the two infinite plays result in sequences $a_0 < a_1 < a_2 < \dots$ and $a_0 < a_2 < a_3 < \dots$. Since Player 0 plays with her winning strategy in both plays. it follows that $X = [0, a_0) \bigcup_{n \in \omega} [a_{2n+1}, a_{2n+2}) \in U$, but also $X' = [0, a_0] \cup \bigcup_{n>0} [a_{2n}, a_{2n+1}) \in U$. By closure under intersection, it follows that $X \cap X' = [0, a_0) \in U$. But U is a free ultrafilter, so it cannot contain a finite set.

It follows by the same argument that Player 1 cannot have a winning strategy. ∎

3 Determinacy by simple strategies

In general, strategies can be very complicated as they may depend on the entire history of a play. However, there are interesting classes of games that are determined via relatively simple winning strategies. We will discuss several kinds of restricted strategies:

(1) **Decomposition invariant strategies** are strategies that depend only on the finite path that has been produced so far, and not on its decomposition into the moves of the players. Thus, a decomposition invariant strategy is a function assigning to each finite path a finite prolongation. We will show that, whenever a player has a winning strategy in a Banach-Mazur game, then she also has one that is decomposition invariant.

(2) **Positional strategies** (also called memoryless strategies) depend only on the current position, and not on the history of the play. On a game graph $G = (V, E)$ a positional strategy is a function $f : V \to V^*$ assigning to every node v a finite path $f(v) \in \text{FinPaths}(G, v)$. It is easy to find determined games that require non-positional winning stategies, but we will prove that all Banach-Mazur games with ω-regular objectives are determined via positional winning strategies.

(3) More generally, **strategies with memory** $\mathfrak{M}$ depend on the history of the play in a restricted way, via a memory structure $\mathfrak{M}$, consisting of a set of memory locations and an update function that changes the memory location as the play proceeds. Strategies with a finite memory structure can be implemented by a finite automaton. We will show that, for Banach-Mazur games, finite memory structures are irrelevant in the sense that winning strategies with finite memory can always be transformed into positional winning strategies. This is in sharp contrast to the usual graph games where already quite simple ω-regular winning conditions (such as, in particular, Muller conditions) lead to games that are determined by finite-memory strategies , but not by positional ones.

3.1 Decomposition invariant strategies

DEFINITION 8. A decomposition invariant strategy in a Banach-Mazur game on a graph (G, v) is a function $f : \text{FinPaths}(G, v) \to \text{FinPaths}(G, v)$ such that $x \leq f(x)$ for all x.

THEOREM 9. *Every Banach-Mazur game that is determined is also determined via a decomposition invariant strategy.*

PROOF. Suppose that Player 1 has a winning strategy for the game $\text{BM}(G, v, \text{Win})$. Then $\text{Win} = \bigcup_{n<\omega} X_n$ with X_n nowhere dense. This means that the complement of each X_n contains a dense open set Y_n. Hence there exists a function g_n assigning to each finite path y a prolongation $g_n(y)$ such that $\mathcal{O}(g_n(y)) \subseteq Y_n$. We define a decomposition-invariant strategy g as follows. Given a finite path $x \in \text{FinPaths}(G, v)$, there are only finitely many $n < \omega$ such that $g_n(y) \leq x$ for some $y \in \text{FinPaths}(G, v)$. Take the minimal n such that this is not the case and set $g(x) = g_n(x)$.

It remains to show that g is a winning strategy for Player 1. Let π be any infinite play that is consistent with g. For every $n < \omega$ there exists a prefix y such that $g_n(y) < \pi$. Hence $\pi \in Y_n$ for all n, which means that π is won by Player 1.

The argument for Player 0 is analogous ∎

3.2 Positional determinacy

To start, we present a simple example of a Banach-Mazur game that is determined, but does not admit a positional winning strategy.

Example 10 Let G_2 be the completely connected directed graph with nodes 0 and 1, and let the winning condition for Player 0 be the set of infinite sequences with infinitely many initial segments that contain more ones than zeros. Clearly, Player 0 has a winning strategy for this game, but not a positional one.

Note that this winning condition is on the $\mathbf{\Pi}_2$-level of the Borel hierarchy. As we show next, this is the lowest level with such an example.

PROPOSITION 11. *If Player 0 has a winning strategy for a Banach-Mazur game with a winning condition* $\text{Win} \in \mathbf{\Sigma}^0_2$*, then she also has a positional winning strategy.*

PROOF. Suppose that Player 0 has a winning strategy f for the Banach-Mazur game $\text{BM}(G, v, \text{Win})$ such that Win is a countable union of closed sets. We have $\text{Win} = \bigcup_{n<\omega}[T_n]$ where each $T_n \subseteq \text{FinPaths}(G, v)$ is closed under prefixes. Further, we can assume that the winning strategy f is decomposition invariant. We claim that, in fact, Player 0 can win with one move, i.e. there is a finite path x such that $\mathcal{O}(x) \subseteq \text{Win}$.

We construct this move by induction. Let x_1 be the initial path chosen by Player 0 according to f. Let $i \geq 1$ and suppose that we have already constructed a finite path $x_i \notin \bigcup_{n<i} T_n$. If $x_i y \in T_i$ for all finite y, then all infinite plays extending x_i remain in Win, hence Player 0 wins with the initial move $w = x_i$. Otherwise choose some y_i such that $x_i y_i \notin T_i$, and suppose that Player 1 prolongs the play from x_i to $x_i y_i$. Let $x_{i+1} := f(x_i y_i)$ the result of the next move of Player 0, according to her winning strategy f.

If this process did not terminate, then it would produce an infinite play that is consistent with f and won by Player 1. Since f is a winning strategy, this is impossible. Hence there exists some $m < \omega$ such that $x_m y \in T_m$ for all y. Thus, if Player 0 moves to x_m in her opening move, then she wins, no matter how the play proceeds afterwards. In particular, Player 0 wins with a positional strategy. ∎

While many important winning conditions are outside $\mathbf{\Sigma}_2^0$, they may well be Boolean combinations of $\mathbf{\Sigma}_2^0$-sets. For instance, this is the case for parity conditions, Muller conditions, and more generally, all ω-regular winning conditions. In the classical framework of infinite games on graphs (where moves are along single edges rather than paths) it is well-known that parity games admit positional winning strategies [6, 17, 9], whereas there are simple games with Muller conditions that require strategies with some memory. We will see that for Banach-Mazur games, the class of winning conditions guaranteeing positional winning strategies is much larger than for classical graph games.

3.3 Banach-Mazur games with Muller winning conditions

A Muller condition is any property of infinite sequences $x \subseteq C^\omega$ that depends only on which symbols $c \in C$ occur infinitely often in x. Muller conditions are of crucial importance in automata theory and in the theory of infinite games. It is one of the standard acceptance conditions for automata on infinite words or infinite trees

DEFINITION 12. A Muller condition on a set C is written in the form $(\mathcal{F}_0, \mathcal{F}_1)$ where $\mathcal{F}_0 \subseteq \mathcal{P}(C)$ and $\mathcal{F}_1 = \mathcal{P}(C) \setminus \mathcal{F}_0$. Given a game graph $G = (V, E)$ whose nodes are labelled by a function $\Omega : V \to C$, a play $\pi \in \text{Paths}(G, v)$ is won by Player σ if, and only if, the set of colours occurring infinitely often on π belongs to $\mathcal{F}_\sigma$.

Usually it is assumed that the set C of colours is finite. In that case there is a precise characterisation, due to Zielonka [24] of the Muller winning conditions that guarantee positional determinacy for the classical form of graph games. It sates that all games with winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ are positionally determined if, and only if, neither $\mathcal{F}_0$ nor $\mathcal{F}_1$ contains a strong split, which means that there do not exist two sets $X, Y \in \mathcal{F}_\sigma$ such that $X \cap Y \neq \emptyset$ and $X \cup Y \in F_{1-\sigma}$.

However, as we show now, *all* Muller conditions (on a finite set of colours) guarantee positional determinacy for Banach-Mazur games.

PROPOSITION 13. *All Banach-Mazur games* $\text{BM}(G, v_0, (\mathcal{F}_0, \mathcal{F}_1))$ *with a Muller winning condition on a finite set of colours are positionally determined.*

PROOF. We write $w \geq v$ to denote that position w is reachable from position v. For every position $v \in V$, let $C(v)$ be the set of colours reachable from v, that is, $C(v) := \{\Omega(w) : w \geq v\}$. Obviously, $C(w) \subseteq C(v)$ whenever $w \geq v$. In case $C(w) = C(v)$ for all $w \geq v$, we call v a *stable* position. Note that from every $u \in V$ some stable position is reachable. Further, if v is stable, then every reachable position $w \geq v$ is stable as well.

We claim that Player 0 has a winning strategy in $\text{BM}(G, v, (\mathcal{F}_0, \mathcal{F}_1))$ if, and only if, there is a stable position w that is reachable from the initial position v, so that $C(w) \in \mathcal{F}_0$.

To see this, let us assume that there is such a stable position v with $C(w) \in \mathcal{F}_0$ for a stable position $w \geq v$. Then, for every $u \geq w$, we choose a path p from u so that, when moving along p, each colour of $C(u) = C(w)$ is visited at least once, and set $f(u) := p$. In case v is not reachable from w, let $f(v)$ be some path that leads from v to w. Now f is a positional winning strategy for Player 0 because, after the first move, no colours other then those in $C(w)$ are seen. Moreover, every colour in $C(w)$ is visited at each move of Player 0, hence, infinitely often.

Conversely, if for every stable position w reachable from v we have $C(w) \in \mathcal{F}_1$, we can construct a winning strategy for Player 1 in a similar way. ∎

Note that in a finite arena all positions of a strongly connected component that is terminal, i.e., with no outgoing edges, are stable. Thus, the above characterisation translates as follows: Player 0 wins the game if, and only if, there is a terminal component whose set of colours belongs to $\mathcal{F}_0$. Obviously this can be established in linear time w.r.t. the size of the arena and a suitable description of the Muller condition.

COROLLARY 14. *On a finite arena G, Banach-Mazur games with a Muller winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ can be solved in time $O(|G| \cdot |\mathcal{F}_\sigma|)$.*

We remark that solving single-step graph games with Muller winning conditions is PSPACE-complete. This is not too difficult to derive from the analysis presented in [5]. A detailed complexity analysis, for a number of different presentations of Muller conditions, can be found in [11].

3.4 Elimination of finite memory

We introduce a general notion of a memory structure and of a strategy with memory. The memory can be finite, as in the finite memory strategies studied for instance in [5], or infinite is in the strategies used in [7].

DEFINITION 15. A *memory structure* for a game graph $G = (V, E)$ is given by a triple $\mathfrak{M} = (M, m_0, \text{update})$, with a set of *memory states* M, an initial state m_0 and a *memory update function* update : $M \times V \to M$. The *size* of the memory is the cardinality of the set M. A *strategy with memory* $\mathfrak{M}$ for a Banach-Mazur game on G is given by a next-move function $f : V \times M \to V^*$ such that $f(v, m) \in \text{FinPaths}(G, v)$ for all $v \in V, m \in M$.

Notice that the local memory update function extends to a function memory : $M \times V^* \to V$, where memory(m, x) is the memory state that is reached by a sequence of updates along a path x, starting with memory state m. This function is defined inductively by

$$\text{memory}(m, \varepsilon) = m, \qquad \text{memory}(m, xv) := \text{update}(\text{memory}(m, x), v).$$

In particular, if a play has gone from initial position v_0 through a finite path $x \in \text{FinPaths}(G, v_0)$ ending at node v, then the memory state is $m = \text{memory}(m_0, x)$, and the strategy defined by $\mathfrak{M}$ and F will prolong x by the path $F(v, m)$.

We will say that a game is determined via memory $\mathfrak{M}$ if one of the players has a winning strategy with memory $\mathfrak{M}$.

THEOREM 16. *A Banach-Mazur game that is determined via a finite-memory winning strategy is in fact positionally determined.*

PROOF. Let us assume that Player 0 wins a Banach-Mazur game on (G, v_0) with a strategy $f : V \times M \to V^*$ based on a finite memory structure $\mathfrak{M} = (M, m_0, \text{update})$. For any node $v \in V$, we denote by $M(v)$ the set of memory locations memory(m_0, x) such that x is path from v_0 to v that may arise as an initial segment of some play consistent with f.

$$M(v) := \{\text{memory}(m_0, x) : x \text{ prolongs } f(v_0, m_0) \text{ and leads to } v\}.$$

Let $\{m_1, m_2, \ldots, m_n\}$ be an enumeration of $M(v)$, in which the initial memory m_0 is taken first, in case it belongs to $M(v)$. We construct paths $y_1 < y_2 < \cdots < y_n \in \text{FinPaths}(G, v)$. First, set $y_1 := f(v, m_1)$. Then, for $1 \leq i < n$, let y_{i+1} be the concatenation of y_i with the path $f(v_i, \text{memory}(m_i, y_i))$ where v_i is the end node of y_i. Finally, set $f'(v) := y_n$.

Clearly f' is a positional strategy. We claim that it is a winning strategy for Player 0. Consider any play π that is consistent with f'. Cleary $f(v_0, m_0)$ is an initial segment of π. Further, suppose that after some finite number of moves, an initial segment x ending at position v has been produced. Player 0 now prolongs x by the path $f'(v)$.

We claim that the path $f'(v)$ can be written in the form $z_1 z_2 z_3$ such that there exist $v' \in V$ and $m' \in M$ with

- z_1 ends at node v'
- memory$(m_0, xz_1) = m'$,
- $z_2 = f(v', m')$

Indeed, if $M(v) = \{m_1, \ldots, m_n\}$, we have memory$(m_0, x) = m_i$ for some $i \leq n$. Let $z_1 := y_i$. Then $v' = v_i$, $m' = \text{memory}(m_0, xy_i) = \text{memory}(m_i, y_i)$, and $f'(v) = z_1 f(v', m') z_3$ for appropriate z_3.

In other words, every move of Player 0 has some "good part" z_2 that would also have been produced by the strategy f if Player 0 had to choose at the position v' with current memeory state m'. But this means that the play cannot be distinguished from a play where Player 0 always moved according to the strategy f while all the "bad parts" were produced by Player 1. Hence the play is also consistent with f and therefore won by Player 0.

The same construction works for Player 1, if we define $M(v) := \{\text{memory}(m_0, x) : x \text{ is a path from } v_0 \text{ to } v\}$. ∎

This result has very interesting consequences for Banach-Mazur games with ω-regular winning conditions. Let $G = (V, F)$ be a game graph with a colouring $\Omega : V \to C$ of the nodes by a finite number of colours and consider winning conditions given by an ω-regular set $W \subseteq C^\omega$. Such conditions can by defined by a formula in some appropriate logic over infinite paths. In the most general case, we have S1S-formulae (i.e. MSO-formulae on infinite paths with vocabulary $\{<\} \cup \{P_c : c \in C\}$). It is well known that every S1S-definable class of infinite words can be recognised by a deterministic Muller or parity automaton (see e.g. [8]). Hence, by a standard construction, any game on a graph G with an ω-regular winning condition can be reformulated as a game on a graph $G \times \mathfrak{M}$, for a finite memory structure $\mathfrak{M}$, with a Muller (or parity) winning condition. This means that we get for G a winning strategy with memory $\mathfrak{M}$ for one of the players. Theorem 16 tells us that in the case of Banach-Mzur games, we can get rid of this finite memory.

THEOREM 17. *All Banach-Mazur games with ω-regular winning conditions are positionally determined.*

4 Definability

We now discuss the question in what logics (MSO, μ-calculus, FO, CTL*, ...) winning positions of Banach-Mazur games with ω-regular winning conditions can be defined. Given any formula φ from a logic on infinite paths (like S1S or LTL), we define the game formula $\Game\varphi$, to be evaluated over game graphs, with the meaning that

$$G \models \Game\varphi(v) \iff \text{Player 0 wins the Banach-Mazur game } \mathrm{BM}(G, v, \varphi).$$

Note that the operation $\varphi \mapsto \Game\varphi$ maps a formula over infinite paths to a formula on graphs. Given a logic L over infinite paths, and a prefix let Game-$L := \{\Game\varphi : \varphi \in L\}$. As usual we write $L \leq L'$ to denote that every formula in the logic L is equivalent to some formula from the logic L'.

Our main definability result can be stated as follows.

THEOREM 18.
(1) Game-S1S $\leq L_\mu$
(2) Game-LTL $\equiv$ Game-FO $\leq$ CTL*.

Obviously, the properties expressed by formulae $\Game\varphi$ are invariant under bisimulation. This has two relevant consequences:

(a) We can restrict attention to trees (obtained for instance by unravelling the given game graph from the start node).
(b) It suffices to show that, on trees, Game-S1S $\leq$ MSO, and Game-FO $\leq$ MPL where MPL is *monadic path logic*, i.e., monadic second-order logic where second-order quantification is restricted to infinite paths.

Indeed, it has been proved by Janin and Walukiewicz [12] that every bisimulation-invariant class of trees that is MSO-definable is also definable in the modal μ-calculus. Similarly, it is known from results by Hafer and Thomas [10] and by Moller and Rabinovitch [16], that every bisimulation invariant property of trees expressible in MPL is also expressible in CTL*.

PROPOSITION 19. *On trees,* Game-S1S $\leq$ MSO.

PROOF. Let $x \leq y$ denote that y is reachable from x. A (decomposition-invariant) strategy for Player 0 in a game BM(T, r, Win) on a tree $T = (V, E)$ with root r is a partial function $f : V \to V$, such that $w < f(w)$ for every w; it is winning if every infinite path through T containing $r, f(r), y_1, f(y_1), y_2, f(y_2) \ldots$, where $f(y_i) < y_{i+1}$ for all i, is contained in Win. An equivalent description can be given in terms of the set $X = f(V)$. A set $X \subseteq V$ defines a winning strategy for Player 0 in the game BM(T, r, Win) if

(1) $(\forall x \in X)\forall y(x < y \to (\exists z \in X)(y < z))$
(2) every path hitting X infinitely often is in Win (i.e. it is winning for Player 0)
(3) X is non-empty.

Clearly these conditions are expressible in MSO. ∎

To deal with winning conditions defined in first-order logic (or equivalently, LTL), we use a normal form for first-order logic on infinite paths (with $<$) that has been established by Thomas [22]. A first-order formula $\varphi(\overline{x})$ is *bounded* if it only contains bounded quantifiers of form $(\exists y \leq x_i)$ or $(\forall y \leq x_i)$.

PROPOSITION 20. *On infinite paths, every first-order formula is equivalent to a formula of the form*

$$\bigvee_i \Big(\exists x(\forall y \geq x)\varphi_i \wedge \forall y(\exists z \geq y)\vartheta_i\Big)$$

where φ_i and ϑ_i are bounded.

THEOREM 21. *On trees,* Game-FO $\leq$ FO*.*

PROOF. Let $\psi = \bigvee_i \Big(\exists x(\forall y \geq x)\varphi_i \wedge \forall y(\exists z \geq y)\vartheta_i\Big)$ be a first-order formula on infinite paths describing a winning condition. We claim that, on trees, $\partial\psi$ is equivalent to the first-order formula

$$\psi^* := (\exists p_1)(\forall p_2 \geq p_1)(\exists p_3 \geq p_2) \bigvee_{i \in I} \psi_i^{(b)} \quad \text{where}$$

$$\psi_i^{(b)} := (\exists x \leq p_1)(\forall y \,.\, x \leq y \leq p_2)\varphi_i \wedge (\forall y \leq p_2)(\exists z \,.\, y \leq z \leq p_3)\vartheta_i.$$

Let $T = (V, E)$ and suppose first that Player 1 has a winning strategy for $\mathrm{BM}(T, r, \psi)$. We prove that $T \models \neg\psi^*$. To see this we have to define an appropriate Skolem function $g : p_1 \mapsto p_2$ such that, for all $p_3 \geq p_2$ and all $i \in I$,

$$T \models \neg\psi_i^{(b)}(p_1, p_2, p_3).$$

Fix any p_1 that we can consider as the first move of Player 0 in the game $\mathrm{BM}(T, r, \psi)$ and any play P (i.e., any infinite path through T) that prolongs this move and that is consistent with the winning strategy of Player 1. Since Player 1 wins, we have that $P \models \neg\psi$. Hence, there exists some $J \subseteq I$ such that

$$P \models \bigwedge_{i \in J} \forall x(\exists y \geq x)\neg\varphi_i \wedge \bigwedge_{i \in I - J} \exists y(\forall z \geq y)\neg\vartheta_i.$$

To put it differently, there exist

- for every $i \in J$ and every $a \in P$, a witness $h_i(a) \in P$ such that $P \models \neg\varphi_i(a, h_i(a))$, and
- for every $i \in I - J$, an element b_i such that $P \models (\forall z \geq b_i)\neg\vartheta_i(b_i, z)$.

Now set

$$p_2 := \max(\{h_i(a) : a \leq p_1, i \in J\} \cup \{b_i : i \in I - J\}).$$

For any p_3 we now obviously have that $T \models \neg\psi_i^{(b)}(p_1, p_2, p_3)$.

For the converse, let $f : V \to V$ be a winning strategy for Player 0 in $\mathrm{BM}(T, r, \psi)$. We claim that $T \models \psi^*$. Toward a contradiction, suppose that $T \models \neg\psi^*$. Hence there exists a Skolem function $g : V \to V$ assigning to each p_1 an appropriate $p_2 \geq p_1$ such that $T \models$

$\neg\psi_i^{(b)}(p_1, p_2, p_3)$ for all $p_3 \geq p_2$ and all $i \in I$. We can view g as a strategy for Player 1 in the game BM(T, r, ψ). If Player 0 plays according to f and Player 1 according to g, then the resulting infinite play $f\hat{\ }g = q_1q_2q_3\ldots$ satisfies ψ (because f is a winning strategy). Hence there exists some $i \in I$ such that

$$f\hat{\ }g \models \exists x(\forall y \geq x)\varphi_i \wedge \forall y(\exists z \geq y)\vartheta_i.$$

Let a be a witness for x so that $f\hat{\ }g \models (\forall y \geq a)\varphi_i(a, y)$. Choose the minimal odd k, such that $a \leq q_k$, and set $p_1 := q_k$. Then $q_{k+1} = g(q_k) = g(p_1) = p_2$. Since $f\hat{\ }g \models \forall y(\exists z \geq y)\vartheta_i(y, z)$, we have, in particular, for every $b \leq p_2$ a witness $h(b) \geq b$ on $f\hat{\ }g$ such that $f\hat{\ }g \models \vartheta_i(b, h(b))$. Choose $p_3 = \max\{h(b) : b \leq p_2\}$. It follows that $f\hat{\ }g \models \psi_i^{(b)}(p_1, p_2, p_3)$. Since $\psi_i^{(b)}$ is bounded, its evaluation on T is equivalent to its evaluation on $f\hat{\ }g$. Hence we have shown that there exists p_1 such that for $p_2 = g(p_1)$, given by the Skolem function g, we can find a p_3 with $T \models \psi_i^{(b)}(p_1, p_2, p_3)$. But this contradicts the assumption that g is an appropriate Skolem function for $\neg\psi^*$.

We have shown that whenever Player 0 has a winning strategy for BM(T, r, ψ) then $T \models \psi^*$ and whenever Player 1 has a winning strategy, then $T \models \neg\psi^*$. By contraposition and determinacy, the reverse implications also hold. ∎

Theorem 18 is implied by Proposition 19 and Theorem 21.

We have seen that for every fixed winning condition expessible in S1S, the winner of the associated Banach-Mazur games is uniformly definable in the μ-calculus. Notice however that this requires that we consider games with a fixed number of local parameters (colours) by which this winning condition is defined. But in the theory of infinite game, a number of algorithmic question concerns classes of games where the number of coulours to define the winning condition is not fixed, but may depend on the game graph.

The most important example is the parity condition: Given a function $\Omega : V \rightarrow \omega$, Player 0 wins those infinite plays in which the least value appearing infintely often is even. For the usual format of graph games, one of the most prominent open problems is the question whether the winning regions of parity games are computable in polynomial time. This problem is equivalent to the question whether the modal μ-calculus admits a polynomial-time model checking algorithm. Even if the range of Ω, i.e. the number of colours, is assumed to be finite, it is not bounded.

For parity games with a fixed number d of colours, which can be viewed as structures $(V, E, P_0, \ldots, P_{d-1})$, it is well-known that the winner is computable in polynomial-time and definable by a μ-calculus formula (with d alternations between least and greatest fixed points). The interesting problem concerns the case of an unbounded number of priorities, and the current algorithms for solving parity games only have upper time-complexity bounds that are exponential in the number of colours.

What about the definability of winning positions in parity games with an unbounded number of colours? First, of all we have to represent the structures in a different way, to avoid an infinite vocabulary. For instance we can describe game graphs as structures

$$(V, E, \prec, \text{Odd})$$

where $u \prec v$ means that u has a smaller colour than v, and Odd is the set of nodes with an odd colour. We denote this class of structures by $\mathcal{PG}$.

The descriptive complexity of parity games, i.e. the question in which logics, winning regions of parity games are definable, has been considered in [4]. The descriptive complexity of a problem provides an insight into the structure of the problem, and the sources of algorithmic difficulty, as the logical resources needed to specify the problem are closely tied to its structure. In the case of parity games, the questions that naturally arise are whether the problem is definable in the least fixed-point logic (LFP) and in monadic second-order logic (MSO), as these are logics with which it is closely associated.

It has been proved in [4] that on arbitrary (finite or infinite) game graphs, parity games are not definable in the least fixed point logic LFP. On finite games graphs, it turned out that the winning regions are LFP-definable if, and only if, they are computable in polynomial-time (despite the fact that, on unordered finite structures, LFP is weaker than PTIME).

Again, it turns out that the analogous question for Banach-Mazur games is simpler.

THEOREM 22. *Winning regions of Banach-Mazur games with the parity winning condition are uniformly definable in least fixed-point logic LFP.*

PROOF. In the proof of Proposition 13 we have shown that Player 0 wins a Banach-Mazur game on (G, v) with a Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ if, and only if, there is a stable position w, reachable from v, such that $C(w) \in \mathcal{F}_0$. For parity games $C(w) \in \mathcal{F}_0$ means that the least colour in $C(w)$ is even. Clearly, this condition is uniformly definable in least-fixed point logic on $\mathcal{PG}$. ∎

This result in fact applies to weaker logics than LFP. It suffices that reachability statements "there is a path from x to y" are expressible. Also, the result may apply to stronger classes of Muller conditions, but it depends on how these are described. In what ever way, such a condition $(\mathcal{F}_0, \mathcal{F}_1)$ is presented on the given game graphs, the necessary condition to express is that $C(w) \in \mathcal{F}_0$.

5 Path games with bounded alternations

Banach-Mazur games have an infinite sequence of alternating moves of the two players. There is an interesting variation of such games where one of the player only makes finitely many moves and eventually one player plays alone. To describe the alternation patterns of such games, we now call the players Ego and Alter, and denote a move where Ego selects a finite path by E, and an ω-sequence of such moves by E^ω; for Alter, we use corresponding notation A and A^ω.

Hence, for any graph G initial position v, and winning condition Win we have the following games.

- $(EA)^\omega(G, v, \text{Win})$ is the ususal Banach-Mazur games with infinite alternation between the two players. By exchanging the roles of the players, we get the game $(AE)^\omega(G, v, \text{Win})$.
- $(EA)^k E^\omega(G, v, \text{Win})$ and $A(EA)^k E^\omega(G, v, \text{Win})$, for arbitrary $k \in \omega$, are the games ending with an infinite path extension by Ego.
- $(AE)^k A^\omega(G, v, W)$ and $E(AE)^k A^\omega(G, v, W)$ are the games where Alter chooses the final infinite lonesome ride.

All these games together form the collection $\mathrm{Path}(G, v, W)$ of *path games*. (Obviously two consecutive finite path moves by the same players correspond to a single move, so there is no need for quantifier strings containing EE or AA.)

Pistore and Vardi [20] used path games of this form for task planning in nondeterministic domains.

5.1 Comparison of path games

For two games $\mathcal{G}$ and $\mathcal{H}$ we write $\mathcal{G} \preceq \mathcal{H}$ if, from the point of view of Ego, $\mathcal{H}$ is better than $\mathcal{G}$. More precisely, $\mathcal{G} \preceq \mathcal{H}$ if, whenever Ego has a winning strategy for $\mathcal{G}$ then he also has one for $\mathcal{H}$, and if Alter has a winning strategy for $\mathcal{H}$ then he has one also for $\mathcal{G}$. Finally, $\mathcal{G} \equiv \mathcal{H}$ if $\mathcal{G} \preceq \mathcal{H}$ and $\mathcal{H} \preceq \mathcal{G}$.

It turns out that this infinite collection of games defined by the game quatifier prefixes over E and A collapses in a uniform way to a finite lattice of just eight different games. This has been observed independently in [3] and [20].

THEOREM 23. *For every arena G and every winning condition* Win*, we have*

$$\begin{array}{ccccccc}
E^\omega(G,v,\mathrm{Win}) & \succeq & EAE^\omega(G,v,\mathrm{Win}) & \succeq & AE^\omega(G,v,\mathrm{Win}) & & \\
 & & \curlyvee\!| & & \curlyvee\!| & & \\
 & & (EA)^\omega(G,v,\mathrm{Win}) & \succeq & (AE)^\omega(G,v,\mathrm{Win}) & & \\
 & & \curlyvee\!| & & \curlyvee\!| & & \\
 & & EA^\omega(G,v,\mathrm{Win}) & \succeq & AEA^\omega(G,v,\mathrm{Win}) & \succeq & A^\omega(G,v,\mathrm{Win})
\end{array}$$

Further, every path game $\mathcal{H} \in \mathrm{Path}(G, v, \mathrm{Win})$ is equivalent to one of these eight games.

PROOF. The comparison relations in the diagram follow by trivial arguments. We just illustrate them for one case. To show that $\mathcal{G} \succeq \mathcal{H}$ for $\mathcal{G} = EAE^\omega(G, v, \mathrm{Win})$ and $\mathcal{H} = (EA)^\omega(G, v, \mathrm{Win})$, consider first a winning strategy f of Ego in $\mathcal{H}$. Ego can use this strategy also for $\mathcal{G}$: he just plays as if he would play $\mathcal{G}$, making an arbitrary move whenever it would be Alter's turn in $\mathcal{H}$. Any play in $\mathcal{G}$ that is consistent with this strategy, is also a play in $\mathcal{H}$ that is consistent with f, and is therefore won by Ego. Second, consider a winning strategy g of Alter in $\mathcal{G}$. In $\mathcal{H} = (EA)^\omega(G, v, \mathrm{Win})$, Alter answers the first move of Ego as prescribed by g, and moves arbitrarily in all further moves. Again, every play that can be produced against this strategy is also a play of $\mathcal{G}$ that is consistent with g, and is therefore won by Alter. In all other cases the arguments are analogous.

To see that any other path game over (G, v, Win) is equivalent to one of those displayed, it suffices to show that

(1) $(EA)^k E^\omega(G, v, \mathrm{Win}) \equiv EAE^\omega(G, v, \mathrm{Win})$, for all $k \geq 1$, and
(2) $A(EA)^k E^\omega(G, v, \mathrm{Win}) \equiv AE^\omega(G, v, \mathrm{Win})$, for all $k \geq 0$.

By duality, we can then infer that $(AE)^k A^\omega(G, v, \mathrm{Win}) \equiv AEA^\omega(G, v, \mathrm{Win})$ for $k \geq 1$ and $E(AE)^k A^\omega(G, v, \mathrm{Win}) \equiv EA^\omega(G, v, \mathrm{Win})$ for all $k \geq 0$.

The equivalences (1) and (2) follow with similar reasoning. Ego can modify a strategy f for $EAE^\omega(G, v, \text{Win})$ to a strategy for $(EA)^k E^\omega(G, v, \text{Win})$. He chooses the first move according to f and makes arbitrary moves the next $k-1$ times; he then considers the entire $A(EA)^{k-1}$-sequence of moves, which were played after his first move, as one single move of A in $EAE^\omega(G, v, \text{Win})$ and completes the play again according to f. The resulting play of $(EA)^k E^\omega(G, v, \text{Win})$ is also consistent with f in $EAE^\omega(G, v, \text{Win})$. Conversely a strategy of Ego for $(EA)^k E^\omega$ also works if his opponent lets Ego move for him in all moves after the first one, i.e., in the game $EAE^\omega(G, v, \text{Win})$. All other equalities are treated in a similar way. ∎

The question arises whether the eight games displayed in the diagram are really different or whether they can be collapsed further. The answer depends on the game graph and the winning condition but for each comparison $\succeq$ in the diagram we find simple cases where it is strict. Indeed, standard winning conditions $\text{Win} \subseteq \{0,1\}^\omega$ on the completely connected graph G_2 with nodes 0 and 1 show that the eight games in the diagram are distinct.

If the winning condition requires a particular initial segment then Ego wins the path games where he moves first and loses those where Alter moves first. Thus, starting conditions separate the left half of the diagram from the right one. Games with reachability conditions and safety conditions separate games in which only one player moves, i.e. with prefix E^ω or A^ω respectively, from the other ones. A game with a Büchi condition is won by Ego if he has infinite control and lost if he only has a finite number of finite moves (prefix ending with A^ω). Similarly, Co-Büchi conditions separate the games which are controlled by Ego from some time onwards (with prefix ending in E^ω) from the others.

5.2 Positional determinacy of ω-regular path games

We have seen that Banach-Mazur games are positionally determined for any ω-regular winning condition. Does this also hold for path games with bounded alternation between the players?

To establish positional determinacy for ω-regular Banach-Mazur games, we first noticed that the reduction of $S1S$ to deterministic parity automata gives us determinacy by finite-memory strategies. In a second step, we proved that for Banach-Mazur games we can eliminate the finite memory, and reduce finite memory strategies to positional ones. The first of these two steps does not depend on the alternation pattern in the game, and therefore also holds for path games with bounded alternation.

PROPOSITION 24. *For any winning condition $\psi \in$ S1S and any game prefix γ, the path games $\gamma(G, \psi)$ admit finite-memory winning strategies.*

However, the reduction from finite memory strategies to positional ones in the proof of Theorem 16 does rely on infinite alternation between the players. For games where the players alternate only finitely often the situation changes. Intuitively, a winning strategy of the solitaire player eventually forms an infinite path which may not be broken apart into finite pieces to serve as a positional strategy.

PROPOSITION 25. *For any prefix γ with finitely many alternations between the players, there are arenas G and winning conditions $\psi \in$ S1S so that no positional strategy is winning in the game $\gamma(G, v, \psi)$.*

PROOF. Consider, for instance, the arena G_2 from Example 10 and a winning condition $\psi \in$ S1S that requires the number of zeroes occurring in a play to be odd. When starting from position 1 Ego obviously has winning strategies for each of the games $E^\omega(G, \psi)$, $AE^\omega(G, \psi)$, and $EAE^\omega(G, \psi)$, but no positional ones. ∎

Nevertheless, these games are positionally determined for one of the players. Indeed, if a player wins a game $\gamma(G, v, \psi)$ that is finally controlled by his opponent, he always has a positional winning strategy. This is trivial when $\gamma \in \{E^\omega, A^\omega, AE^\omega, EA^\omega\}$; for the remaining cases EAE^ω and AEA^ω a positional strategy can be constructed as in the proof of Theorem 16.

Finally we consider winning conditions that do not depend on initial segments. We say that ψ is prefix independent, if, for any ω-word π and any finite words x and y, we have $x\pi \models \psi$ if, and only if, $y\pi \models \psi$.

THEOREM 26. *For any prefix-independent winning condition $\psi \in$ S1S and every γ, the games $\gamma(G, v, \psi)$ admit positional winning strategies.*

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