Book Review

Gamma

Reviewed by Dan Segal

Gamma
Julian Havil
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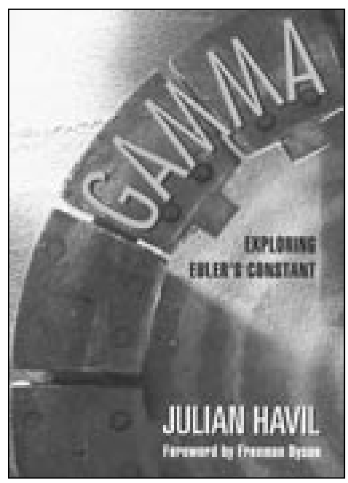
This book is evidently a labour of love. The author, Dr. Julian Havil, is a mathematics teacher at one of the grand old English public (= private) schools. It is a pleasant surprise to see that such teachers still exist: every now and then we see official reports lamenting the state of mathematics education in Britain, but too little is done to make schoolteaching once again a tempting career choice for bright and enthusiastic mathematics graduates (let alone Ph.D.'s).

There has been a recent spate of popular books about mathematics. Most of them assiduously avoid the use of mathematical notation, out of deference to Hawking's exponential decay law "n equations $\Rightarrow 2^{-n}\times$ (readership)". Havil's book, by contrast, positively revels in equations, formulae, tables of numbers and graphs; it is not a book *about* mathematics but a book *of* mathematics. It is genuinely a "popular" book nonetheless, written in an engagingly enthusiastic style and aimed at a specific class of reader. The author neither talks down nor goes over this reader's head: he assumes that you are comfortable with basic calculus, infinite series, etc., and on that basis gets on with telling his story in clear and straightforward language. Actually, it would be more accurate to say "stories", since this is something like a picaresque novel; the hero, Euler's constant γ, serves as the unifying motif through a wide range of mathematical adventures. The book differs from many others in the large proportion of genuine 100-proof maths: there is a particular kind of satisfaction, familiar to us and a mystery to nonmathematicians, that comes with really grasping a proof, and the reader is given plenty of such moments. Of course, not all the results stated can be proved within the constraints of such a book as this, but the author takes care never to blur the distinction. When reproducing some of Euler's more cavalier arguments, for example, he points out that they are not rigorous but can easily be made so: the (correct) implication being that they could be made rigorous by the educated reader without introducing further new ideas. In other words, the reader is given the compliment of being addressed as a professional mathematician.



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The first chapter sets the historical scene in the sixteenth century. We are reminded of the real difficulties faced by astronomers such as Tycho Brahe, who needed to perform large arithmetical calculations, and are given a lively anecdotal account of various labour-saving techniques developed for the purpose. The invention of logarithms is described in some detail; it was something of a revelation, to this reader anyway, to see the contortions that John Napier had to go through to give a "kinetic" definition of his logarithm before the modern concept of a function was around to make everything seem straightforward. This is contrasted with Euler's 1770 definition, the one familiar to us today. The relation between the two is derived at some length (it is not obvious!):

$$\mathrm{NapLog}(x) = 10^7 \log_{1/e}(10^{-7}x);$$

the slightly odd scaling factors in effect allow the writing of 7-figure log tables without decimal points, a notation that was only coming into use around Napier's time.

This chapter is the only one that is primarily historical. It gives an educational glimpse into the mind-set of a much earlier generation of mathematicians—very interesting but requiring some effort to follow mathematically. After this, the emphasis is on the mathematics. The originators are introduced with more or less colourful biographical snapshots, but the ideas are clearly explained in modern notation.

Chapter 2 introduces the harmonic series $\sum \frac{1}{n}$ and its partial sums

$$H_n = \sum_{r=1}^{n} \frac{1}{r}.$$

The author first points out the exceedingly slow growth of these numbers: the least n for which $H_n \geq 100$ is a 44-digit number (taking nearly a whole line of text); he then shows that the series does nonetheless diverge, giving not only the usual argument by grouping of terms but also an analytic proof which evaluates

$$\int_{-\infty}^{0} \frac{e^x}{1-e^x}.$$

Further interesting titbits: H_n is never an integer; indeed except for $n = 1, 2, 6$ the decimal expansion of H_n is nonterminating. While the first fact is quite elementary, the second is reduced to "Bertrand's postulate" (which is explained but not proved). I mention these details because they typify the author's approach, which is carefully designed (1) to arouse the reader's curiosity in a topic, (2) to instill an awareness of how remarkable the true facts often turn out to be, and (3) to demonstrate how often the answer to one question can generate a host of new directions for research.

A similar pattern governs the arrangement of the book as a whole. There is a leading thread of argument concerning relations between the harmonic series, the logarithmic function, and the Gamma function, the emphasis later shifting to the zeta function. This is interspersed with numerous digressions as the author picks up on a point that catches his interest and follows it wherever it may lead. For example, writing

$$\gamma = \lim_{n\to\infty} (H_n - \ln(n+\alpha)),$$

we may ask what choice of α makes this limit converge as fast as possible: a sensible question from the point of view of evaluating γ as a decimal. It is easy enough to see that $\alpha = 0$ gives an error at the nth step of about $\frac{1}{2n}$; some hefty (though as always elementary) calculation shows that $\alpha = \frac{1}{2}$ gives a much better error of about $\frac{1}{24n^2}$. This is used to introduce a whole chapter on the Euler-Maclaurin summation formula, where we get a full asymptotic expansion for γ in terms of Bernoulli numbers, taking in on the way Stirling's formula for $n!$. An earlier chapter introduces the Gamma function. Here we see not only the familiar identity $\Gamma(n) = (n-1)!$ but the more startling formulae $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and

$$\frac{\Gamma'(n)}{\Gamma(n)} = H_{n-1} - \gamma.$$

Results like these accumulate to give the reader a sense of the deep magic of numbers: the hidden threads that tie together familiar yet apparently unrelated things.

The middle part of the book treats us to a couple of chapters of lighter relief, covering a wide range of topics where the harmonic series and/or the logarithmic function make an appearance. These are too many and varied to list here; all are interesting and many quite surprising, such as "Benford's law" governing the statistical distribution of digits in apparently-but-not-quite random lists of numbers.

The last two chapters focus in on the Prime Number Theorem and the Riemann Hypothesis. At the age of fifteen, by studying tables of prime numbers, Gauss made the extraordinary observation that the number $\pi(x)$ of primes less than x seems to approach $\frac{x}{\ln x}$ as x increases without limit; it took more than a century to turn Gauss's insight into the Prime Number Theorem:

$$\pi(x) \sim \frac{x}{\ln x}.$$

Havil builds up to the PNT with a mixture of history and heuristics, including plenty of tables and graphs; by the end of Chapter 15 the reader has a good intuition of the nature of the problem and of its subtlety.

The final chapter bravely attempts to explain the role of the Riemann zeta function, von Mangoldt's "explicit formula" for primes, and its relevance for PNT. The zeros of zeta are discussed at length. This chapter is the most sophisticated mathematically. Of course the main results are not proved, but one gets a clear understanding of what they mean and some feeling for the significance of RH. As the necessary complex analysis is beyond the basic knowledge assumed of the reader, the author provides a crash course (without many proofs) in Appendix D. This is too brief to serve as a source for really learning this material, but there is just enough to make sense of the preceding chapter and enough (for the determined student) to understand the proof of analytic continuation and the functional equation of the zeta function, given in Appendix E.

As far as I could see, there are no genuine mathematical errors in the book nor, impressively, typos in the many formulae. But there are a few unfortunate and curious spots where the author has clearly written something different from what he meant: Table 16.1 puts the early zeros of zeta on the line $Im(z) = \frac{1}{2}$ instead of $Re(z) = \frac{1}{2}$, and this is reproduced in a couple of places in the written text. The boxed statements of Cauchy's Integral Theorem and Integral Formula, in Appendix D, are both wrong, though correctly explained in the surrounding text.

My final two years at (high) school were largely devoted to training in the manipulation of series, integrals, and trigonometric identities. While any skills I once possessed in such matters have long since atrophied, it was a pleasure to be shown in this book how this bag of tricks can be moulded into, and often arose out of, beautiful and deep mathematics. Of course we did not see much of that beauty at school; having been seduced by the simple pleasures of abstract algebra at university, I was more than happy to put integrals and series behind me. But it might have been different if I had been given *Gamma* to read: any bright 18–20-year-old with good training in calculus will not only be able to understand everything in the book but is likely to be inspired by it and want to learn more. Indeed, anyone who knows a bit of calculus (and is not a professional number theorist or analyst) will enjoy this carefully planned tour through five centuries of mathematics.

The relative beginner may find some of the solid analysis a bit daunting but hopefully will be encouraged to persevere by the engaging anecdotal style. As an aging mathematician having a nodding acquaintance with analytic number theory, I found much in the book quite illuminating. Things I had been vaguely aware of before—the Gamma function, the Riemann Hypothesis—now feel more like old friends. I would heartily recommend *Gamma* as a Christmas present for any student of mathematics or for any scientist or engineer who might be pleasantly surprised to find that there is more to pure mathematics than abstract pie-in-the-sky.

The only similar book that I know is *From Fermat to Minkowski*, by Scharlau and Opolka (Springer, 1984). While also interweaving history with number theory, this is much harder mathematically: a reader who has caught the bug from *Gamma* and is willing to work seriously at mathematics would enjoy moving on to the earlier book at a slightly later stage in his or her education.