Review [1] of

**The Square Root of 2:**

**A Dialogue Concerning a Number and a Sequence**

**Author: David Flannery**

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$25.00, Hardcover

**Reviewers: William Gasarch, Alexander Kruskal, Justin Kruskal, Rebecca Kruskal**

Professor Clyde Kruskal, who works in parallelism, has triplets: Alexander, Justin, and Rebecca Kruskal. They are in 9th grade, have already taken elementary algebra, and are taking Geometry now (this was written in Spring 2006). Professor William Gasarch spoke to them about the square root of 2. What follows is an *interpretation* of their conversation.

## Act I: The irrationality of $\sqrt{2}$

**Gasarch:** What are the most important numbers in mathematics?
**Alexander:** $\pi$.
**Rebecca:** I was going to say that!!
**Justin:** I've seen the number $e$ on a calculator.
**Gasarch:** You are missing some easier ones.
**Rebecca:** Zero.
**Justin:** But that's just nothing (all laugh).
**Gasarch:** Rebecca is right— 0 is an important number. And there is one more that's important.
**Alexander:** One.
**Justin:** What about negative one?
**Gasarch:** OH, yes, that's important also.
**Rebecca:** But you said there was just one more left!! If I had known there were two I would have gotten that one!!
**Gasarch:** I'm glad I don't have kids of my own. Now, there is also $i$ which is the square root of $-1$ which we won't go into today, but you may see later in High School. There have been books written about $0, \pi, e$, and $i$.
**Rebecca:** What a boring life that is to write a book on a number.
**Gasarch:** You are both right and wrong.
**Rebecca:** Huh?
**Gasarch:** You are wrong because these books embody math and history of interest. But you also right because, as the review by Brian Blank indicates, the books aren't that good. However, you are wrong because these people did not spend their entire lives on these books.
**Rebecca:** (Confused) So am I right or wrong!!?
**Alexander:** (teasing) You're wrong! You're wrong!
**Gasarch:** Knock it off. OKAY, let me tell you why we are here. There have been books written about 0, $e$, $\pi$, $i$. My plan was to review these books for my column. But I found a marvelous review of the $0, e, \pi$ and $i$ books already, which I got permission to reprint. I then got a book on

$\sqrt{2}$. It seems more pitched to High School Students so I decided to read it, talk to you about it, and see if it really is good for High School Students.
**Justin:** Most high school students won't care about $\sqrt{2}$.
**Gasarch:** The three of you like math, so the question really is "will high school students who like math like this book?" Anyway, after we talk about it I'll essentially write down what we all said and that will be the review, and you'll all be co-authors.
**Rebecca:** Cool!
**Gasarch:** So, lets start. Consider a square that is 1 by 1 (draws on board). What is the length of the diagonal (draws the diagonal).
**Justin:** You use that theorem we learned. Some Geek Guy.
**Gasarch:** I think you mean some Greek guy. His name was Pythagoras.
**Alexander:** You do that square thing. You know, $a^2 + b^2 = c^2$.
**Gasarch:** Yes, (does the algebra on the board) so we get that the this length is such that, if you square it, you get 2.
**Rebecca:** Lets see, $1^1 = 1$ and $2^2 = 4$, so there is no such number.
**Justin:** You're forgetting about fractions.
**Rebecca:** Oh yeah. (Tries some fractions.) None of these work. There is no such number.
**Gasarch:** But I have on the board a line segment that has that length.
**Alexander:** It's irrational.
**Gasarch:** OH, you are saying that there is no fraction with this property. Did you learn that somewhere.
**Alexander:** I think I heard it someplace.
**Gasarch:** Can you prove it?
**Alexander:** No, but isn't it one of those things that everyone kind of knows is true but nobody has proven? Like that the primes are infinite.
**Gasarch:** (not quite sure how to respond) Uh, lets go around and see what you think about the following two questions: Is there a fraction what when squared it is 2, and in either case, is there a proof?
**Justin:** I think that $\sqrt{2}$ is irrational and the proof is really, really, really, really hard.
**Rebecca:** I think that $\sqrt{2}$ is irrational and nobody knows how to prove it.
**Alexander:** I think that $\sqrt{2}$ is irrational and nobody knows how to prove it.
**Rebecca:** Copycat!
**Gasarch:** Why do you think it's hard to prove?
**Justin:** No matter how many places you compute, you'll never know if $\sqrt{2}$ will stop. If it does stop, you'll know it's rational. If it hasn't stopped yet, you don't know whether it's going to, and you can't just compute the whole infinite number of places and then note that it didn't stop. (EDITORIAL NOTE: We then discussed that even if a decimal expansion goes on forever it may still be a rational. We omit this from the review as it is not relevant to the book.)
**Gasarch:** Back to the topic at hand. You all think that $\sqrt{2}$ is irrational and the proof is either hard or unknown. It turns out that you are right that $\sqrt{2}$ is irrational, but actually the proof is easy enough that I can show it to you right now. (Gasarch does the proof on the blackboard and they understand it.)
**Gasarch:** You all thought it would be hard to prove, but it was not. So, why did you think so, and what was the error in your thinking?
**Justin:** This is the first time we saw a proof that something could not be done.

**Alexander:** Actually, this is the first time we saw a proof at all! In Geometry they have these dumb 'two-column proofs' which don't seem like proofs at all.
**Gasarch:** Realize that a proof is a chain of reasoning. The actual form is not important. The proofs you've seen in Geometry are chains of reasoning, though they may not seem that way. Alexander- you had referred to things that "everyone kind of knows are true but there is no proof". There are very few of these things. But more important is, do you find the result and the proof interesting? And be honest— the point of this meeting is to see if this would interest high school students who already like math.
**Justin:** Yes, it was kind of cool to see that you could prove there is no fraction for $\sqrt{2}$.
**Alexander:** Yes. Its interesting to prove that you can't do something.
**Rebecca:** Yes, it's good to see a proof so that we know things are true.
**Gasarch:** The book we are reviewing does not have this proof until page 42.
**Alexander:** What do they do until that point?
**Gasarch:** The book is written as a conversation between a professor and a student. So their conversation was much like ours, discussing $\sqrt{2}$, having the student try to find a rational number for it, etc.
**Rebecca:** But 42 pages. That sounds really boring, though I like reading books in conversation form.
**Gasarch:** That's the problem I have with the book. I'm not sure who wants to READ the book. For someone like me the book is too elementary. For someone like you the book may be too boring in that they spend a long time getting anywhere. On the other hand, I did learn some stuff in there that is worth telling you.

## Act II: Tiling Problems

**Alexander:** What else is there to even say about $\sqrt{2}$? It's just a number.
**Gasarch:** Lets go to a different problem. Say you have a $\sqrt{2}$ by 1 rectangle (Draws on board). Can you cover it with squares that are $x$ by $x$. Such a covering is called a tiling.
**Justin:** Yes you can- just make $x$ small enough.
**Rebecca:** No you can't.
**Alexander:** Yes you can.
**Rebecca:** Can I change my vote? I want to say yes you can.
**Gasarch:** Yes you can change your vote. Now, I'll go around and ask you each again, and ask for your reasons. Justin.
**Alexander:** Once the square is small enough, of course you can make it all work out. Its obvious.
**Justin:** Yeah. What he said.
**Rebecca:** Yeah. What he said.
**Gasarch:** It turns out that you can't. And I'll prove it to you. (The proof is done on the blackboard and they understand it. Gasarch then showed them that every rectangle that has rational sides can be tiled.)
**Rebecca:** You two made me change my vote!! I would have been right if it you didn't make me change my vote!!
**Gasarch:** Rebecca, the point of these votes is to get you thinking about stuff. If you vote wrong and then see what the truth is you are enlightened.
**Alexander:** What if you vote correctly? Are you still enlightened?

**Gasarch:** Yes, but in a different way. You are enlightened to know why you were right. Now back to Tiling- what to you think of the fact that you can't tile.
**Justin:** Wow. So you can't tile. What can you tile?
**Gasarch:** If the sides are rational then you can tile it. (We all do the proof of one case together- I assign the general case for Homework.)

### Approximations to $\sqrt{2}$

**Gasarch:** The book contains a table that looks like this

| $n$ | $n^2$ | $2n^2$ |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 4 | 8 |
| 3 | 9 | 18 |
| 4 | 16 | 32 |
| $\vdots$ | $\vdots$ | $\vdots$ |

We can use this table to approximate $\sqrt{2}$.
**Rebecca:** How?
**Gasarch:** Can there be a number that appears in the second and third columns?
**Rebecca:** Sure.
**Alexander:** Yeah, why not.
**Justin:** Yeah. We just haven't found it yet. Who cares?
**Gasarch:** Actually you cannot (Does the proof).
**Rebecca:** Okay, so you can't. Who cares?
**Gasarch:** If we found two numbers that are the same, that would give us that $\sqrt{2}$ is rational. But if we found two numbers in the table that are very close together, that would give us a very good approximation to $\sqrt{2}$. I want you to find numbers in the second and third column that differ by one.
**Rebecca:** 8 and 9.
**Alexander:** 49 and 50.
**Justin:** 288 and 289.
**Gasarch:** And we can use these to get the following good approximations to $\sqrt{2}$. (After some writing on the board.) We now see that
$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}$
form better and better approximations to $\sqrt{2}$. This sequence has a pattern to it that will enable you to find the next element. Can you figure out the pattern? (After several tries they do not figure it out, though Rebecca claims she had it and lost it. I leave it to the reader to discover a recurrence for the numerator and denominator.)
**Rebecca:** How much of the book have we covered.
**Gasarch:** To be fair, they do more formal proofs and they also show that the fractions generated by this recurrence are in lowest form and are also all of the numbers you ever get by looking at the columns and seeing when they are 1 apart. And they also show that even if you didn't start with $\frac{1}{1}$ the sequence would still approximate $\sqrt{2}$.
**Alexander:** But isn't all of that obvious?

**Gasarch:** No, these things require careful proof. Remember that a while back it was obvious to you that you could tile the 1 by $\sqrt{2}$ rectangle with small enough tiles. And you were wrong.
**Justin:** So does the book teach the need for proof?
**Gasarch:** Not really, but they do proofs.

**Continued Fractions**

**Gasarch:** Note the following: $\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}}$
**Alexander:** (does the algebra) Yes, that's true.
**Gasarch:** Now note that I have an equation that has $\sqrt{2}$ in terms of itself. SO, we can plug in the expression for $\sqrt{2}$ and get
$\sqrt{2} = 1 + \frac{1}{2+\frac{1}{1+\sqrt{2}}}$
**Alexander:** Is that really a fraction?
**Gasarch:** Yes it is. And I can do it again to get

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{1+\sqrt{2}}}}.$$

**Justin:** That's a really weird looking fraction.
**Gasarch:** I can keep doing this forever to get

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{2+\cdots}}}.$$

**Alexander:** What if you get tired of this and stop working it out?
**Gasarch:** Glad you asked! What if you just ignored alot of the terms? What if you just do

- $1$
- $1 + \frac{1}{2}$
- $1 + \frac{1}{2+\frac{1}{2}}$
- $\vdots$

**Rebecca:** We get $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}$.
**Gasarch:** Does the pattern look familiar?
**Alexander:** Yes, it's that other pattern we got.
**Gasarch:** Right. So two different ways to approximate the $\sqrt{2}$ end up being the same way.
**Justin:** Does the book do that?
**Gasarch:** Yes, in fact the book has three sequences that end up being the same that all approximate $\sqrt{2}$. Note that the book's title is "The square root of 2: A Dialogue concerning a number and a sequence." The sequence is that sequence.

**Conclusions**

**Gasarch:** We did four topics today: (1) $\sqrt{2}$ is irrational, (2) tiling, (3) approximations for $\sqrt{2}$, (4) Continued fractions, which lead to more approximations. Did you find these topics interesting.

**Rebeeca:** Yes, but not enough to like, you know, read.
**Alexander:** It was good to see some proofs not in that stupid form we do in Geometry.
**Justin:** I liked those weird fractions. But 42 pages to get to $\sqrt{2}$ being irrational? That seems so weird.
**Alexander:** But if it wasn't for the book we wouldn't be here listening to you talk about this stuff.
**Rebecca:** How long was the book? (she picks it up). 242 pages! That's too long. How much of it did you cover today?
**Gasarch:** I covered about half of it. Probably less if you want to think about how much rigor I left out.
**Justin:** So ... is it a good book or not?
**Gasarch:** The book is a good place to get ideas on what to tell high school students about the $\sqrt{2}$ and is a launching point for many topics of interest. Some High School students might like to read it themselves, but it may be too slow paced. Reading this review may be just as good.