ROTH'S THEOREM ON PROGRESSIONS REVISITED

By

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§0. Statement and discussion of the argument

This paper is a sequel to [B]. Our main result is an improvement of the density condition for a subset $A \subset \{1, \dots, N\}$ to contain a nontrivial arithmetic progression of length 3. More specifically, we prove the following

Theorem 1. *If* $A \subset \{1, \dots, N\}, |A| = \delta N$ *and*

$$\delta \gg \frac{(\log \log N)^2}{(\log N)^{2/3}} \tag{0.1}$$

(N assumed sufficiently large), then A contains nontrivial progressions of length 3.

Recall that the condition required in [B] is

$$\delta \gg \Big(\frac{\log \log N}{\log N}\Big)^{1/2}. \tag{0.2}$$

Roth's original argument [R] assumes

$$\delta \gg \frac{1}{\log \log N}; \tag{0.3}$$

this was subsequently improved by Heath–Brown [H-B] and Szemeredi to

$$\delta < \frac{1}{(\log N)^c} \tag{0.4}$$

for some $c > 0$ ($c = 1/20$ in [Sz]).

The main technical advance in [B] is the increment of the density of A in 'Bohr sets' rather than arithmetic progressions. In [B], we introduce a notion of 'regular Bohr set' and developed a variant of the usual circle method approach in this context.

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Very schematically, the condition (0.2) can be explained as follows. Following the circle method as in Roth's original argument, the density of A in a suitable Bohr set Λ_α is increased from δ_α to $\delta_{\alpha+1}$, where

$$\delta_{\alpha+1} > \delta_\alpha + c\delta_\alpha^2 \tag{0.5}$$

(the Bohr sets Λ_α being the replacement for a decreasing sequence of subprogressions P_α of $\{1, \dots, N\}$, $|P_{\alpha+1}| > |P_\alpha|^{1/2}$).

The passage from Λ_α to $\Lambda_{\alpha+1}$ involves adding one generator. Thus, basically, $\Lambda_\alpha = \Lambda_{\theta^{(\alpha)}, \varepsilon^{(\alpha)}, M_\alpha}$ involves some element $\theta^{(\alpha)}$ in $\mathbb{T}^{d_\alpha}$, $d_\alpha = \alpha$, and consists of the integers $n \in \mathbb{Z}, |n| \leq M_\alpha$ satisfying

$$\|n\theta_j^{(\alpha)}\| < \varepsilon^{(\alpha)} \quad \text{for } j = 1, \dots, d_\alpha. \tag{0.6}$$

Here $\varepsilon^{(\alpha)} \in \mathbb{R}_+$ is a decreasing parameter. We can take

$$\varepsilon^{(\alpha+1)} = \frac{1}{(\log N)^{10}} \varepsilon^{(\alpha)}. \tag{0.7}$$

Also,

$$M_{\alpha+1} < \frac{M_\alpha}{(\log N)^{10}}. \tag{0.8}$$

Since (0.5) requires essentially $1/\delta$ steps to reach density $0(1)$ and, crudely speaking,

$$|\Lambda_\alpha| \gtrsim [\varepsilon^{(\alpha)}]^{d_\alpha} M_\alpha, \tag{0.9}$$

it is clear from the preceding that the condition

$$\log N \gg \frac{1}{\delta} \frac{1}{\delta} \log\log N$$

should be imposed. This explains (0.2).

Let us point out that in the purely algebraic setting, considering (say) subsets $A \subset \mathbb{F}_3^n$, our Bohr sets Λ_α are simply subgroups and $|\Lambda_{\alpha+1}| = \frac{1}{3}|\Lambda_\alpha|$. The condition on $\delta = \frac{|A|}{3^n}$ thus becomes

$$\delta \gg 1/n \tag{0.10}$$

(which would correspond to $\delta \gg 1/\log N$ in the arithmetic setting).

Improving on the condition (0.10) in the algebraic context is a most interesting problem that most likely requires a major new idea.

Returning to $A \subset \mathbb{Z}$ and (0.1), we again rely on the Bohr set technology of [B]. Sections 1–6 in this paper are, in fact, just a repetition of [B]. The only difference is that we need more general Bohr sets $\Lambda_{\theta,\varepsilon,M}$ where $\theta \in \mathbb{T}^d$, $\varepsilon \in \mathbb{R}^d_+$, defined as

$$\Lambda_{\theta,\varepsilon,M} = \{n \in \mathbb{Z} : |n| \leq M \text{ and } \|n\theta_j\| < \varepsilon_j \text{ for } j = 1, \dots, d\} \tag{0.11}$$

(instead of $\varepsilon_j = \varepsilon$). But dealing with this variant does not require any significant changes, and the setup reached in Section 6 is the same as in [B].

Section 7 corresponds to Section 8 from [B]. In fact, a slightly more careful check of the argument in [B] shows that we may actually achieve a density increment $\sim \delta_1$ (rather than $\sim \delta_1^2$) under that particular assumption (see (7.1) in the paper). The novelty here is to provide a more elaborate treatment of the alternative considered in Section 7 of [B] (see Sections 8–10 here). Let us describe the main ideas in a simplified way. Our exponential sums associated to A are of the form

$$S_A(x) = \sum_{n \in A} \lambda_n e^{2\pi i n x}, \tag{0.12}$$

where

$$\lambda_n = \begin{cases} \frac{1}{|\Lambda|} & \text{if } n \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

and Λ is some Bohr set.

We are in the situation that

$$\|S_A - \lambda(A)S\|_\infty = \tau\delta_1, \tag{0.13}$$

where $\tau > \delta_1$, $\lambda(A) = \sum_{n\in A} \lambda_n = \delta_1$ and $S(x) = \sum_{n\in\mathbb{Z}} \lambda_n e^{2\pi i n x}$. The increment (0.5) corresponds to the worse case scenario that $\tau \sim \delta_1$. Let us assume so (the optimal dichotomy as worked out in this paper turns out to distinguish the cases $\tau \gtrsim \delta_1^{1/2}$ and $\tau \lesssim \delta_1^{1/2}$).

Instead of considering a single point $\theta \in \mathbb{T}$ with

$$|S_A(\theta) - \lambda(A)S(\theta)| > \delta_1^2 \tag{0.14}$$

to perform the density increment (as in [B]), consider the entire level set

$$\mathcal{F} = \{\theta \in \mathbb{T} : \ |S_A(\theta) - \lambda(A)S(\theta)| > \delta_1^2\}. \tag{0.15}$$

With a slight twist of things for the sake of simplicity, let us assume not only that $\mathcal{F} \neq \phi$, but that also $\mathcal{F}$ carries a fraction of the L^2-mass of S_A. Thus

$$\int_{\mathcal{F}} |S_A|^2 \sim \int |S_A|^2 = \frac{\lambda(A)}{|\Lambda|} = \frac{\delta_1}{|\Lambda|} \tag{0.16}$$

(the true situation encountered is a bit different — several Bohr sets need to be involved — but this is inessential for this discussion).

Along the lines of [B1] or [Ch], consider a maximal ‘independent’ subset $\psi_1, \dots, \psi_R$ of $\mathcal{F}$. Thus all elements of $\mathcal{F}$ are ‘well approximable’ by sums of the form $\sum_{s=1}^{R} \ell_s \psi_s$, where $\ell_s \in \{0, 1, -1\}$. We distinguish 2 cases. If R is small (in particular, $R \ll \log N$), then we may reduce our Bohr set Λ to a new Bohr set $\bar{\Lambda}$ by adding R generators and reaching density $\bar{\lambda}(A) = |A \cap \bar{\Lambda}|/|\bar{\Lambda}| \sim 0(1)$. In this situation, we are done.

If R is large, we proceed differently. Our aim is to find a (large) subset $I \subset \{1, \dots, R\}$ such that we may introduce a new Bohr set $\bar{\Lambda} \subset \Lambda$ of dimension $d + |I|, d = \dim \Lambda$, in such a way that the density increment $\bar{\lambda}(A) - \lambda(A)$ is at least $|I|\delta_1^2$. Such an increment would have been obtained as well in $|I|$ steps, adding each time 1 generator and increasing the density by δ_1^2. However, obtaining the increment $|I|\delta_1^2$ in a single step is advantageous with respect to the size of the new Bohr set $\bar{\Lambda}$, since now

$$|\bar{\Lambda}| \gtrsim \frac{|\Lambda|}{(\log N)^{10(d+|I|)}}, \tag{0.17}$$

while the iterative process would give us essentially

$$|\bar{\Lambda}| \gtrsim \frac{|\Lambda|}{(\log N)^{10(d+|I|)|I|}}. \tag{0.18}$$

The reason comes from (0.7), which is applied at each step. It should now also be clear to the reader why the more general Bohr sets (0.11), with ε_j depending on the generator θ_j, are necessary with this more refined approach.

There is a formal similarity with the main idea in [HB] to improve over Roth’s method by involving many points rather than a single point at each step. What is different here is, of course, that we work with Bohr sets rather than progressions, but also that a different strategy is applied depending on the structure of the level set $\mathcal{F}$ in (0.15). From the harmonic analysis point of view, the new input appears in our Section 10 and in the Appendix, which analytically speaking are the most interesting parts of the paper.

Finally, it is clear that the novelties introduced have other applications in combinatorial number theory. In particular, one may expect to improve the dependences on the ‘additive doubling constant’ K for the sumset inequality

$$|A + A| \le K|A| \quad (A \subset \mathbb{Z}) \tag{0.19}$$

in M. Chang’s quantitative Freiman theorem [Ch], following T. Sanders’ approach in the $\mathbb{F}_2^n$-setting (see [San]).

More precisely, one may establish the following (see Sanders’ Appendix)

Theorem 2. *Let $A \subset \mathbb{Z}$ be a finite set satisfying $|A+A| < K.|A|$. Then A is contained in a generalized arithmetic progression P of dimension $d \leq [K-1]$ and size*

$$|P| < |A| \exp\{C.(\log K)^C.K^{7/4}\} \tag{0.20}$$

(where C is some constant).

Thus, the improvement over [Ch] lies in the exponent 7/4 in (0.20) (rather than 2).

Following essentially the same argument as in [Ch] (or [TV]), Theorem 1 is derived from

Theorem 3. *Under the assumptions of Theorem 1, the sumset $2A-2A$ contains a Bohr set B on $d' \leq K^{3/4}(\log K)^C$ generators and such that*

$$|B| > |A|. \exp\{-C(\log K)^3.K^{3/4}\}. \tag{0.21}$$

The proof of Theorem 2 is, roughly speaking, an adaptation of [San] ($\mathbb{F}_2^n$ is now replaced by $\mathbb{Z}$) invoking the Bohr-set technology and the method described in this paper to improve density (note that in the $\mathbb{F}_2^n$-context of [San], the exponents $7/4$ in (0.20) and $3/4$ in (0.21) become $3/2$ and $1/2$, resp.; the reason for the weaker result obtained in the $\mathbb{Z}$-case relates again to the earlier discussion on Bohr-sets vs. subspaces).

§1. Definitions

Let $\theta \in \mathbb{R}^d, d \geq 1, \varepsilon_j > 0, N$ a positive integer. Define

$$\Lambda_{\theta,\varepsilon,N} = \{n \in \mathbb{Z} : |n| \leq N, \|n\theta_j\| < \varepsilon_j \text{ for } j = 1,\dots,d\} \tag{1.1}$$

and $\lambda_{\theta,\varepsilon,N} = \lambda$, where

$$\lambda(n) = \begin{cases} |\Lambda_{\theta,\varepsilon,N}|^{-1} & \text{if } n \in \Lambda_{\theta,\varepsilon,N} \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Thus λ is probability measure on $\mathbb{Z}$.

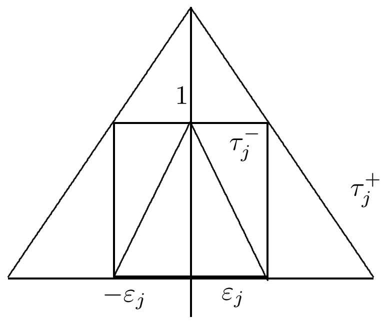
§2. Estimates on Bohr Sets

Lemma 2.0.

$$\text{(i)} \qquad |\Lambda_{\theta,\varepsilon,N}| > \frac{1}{2}\varepsilon_1 \cdots \varepsilon_d N \tag{2.1}$$

$$\text{(ii)} \qquad |\Lambda_{\theta,\varepsilon,N}| < 8^{d+1}|\Lambda_{\theta,\frac{\varepsilon}{2},\frac{N}{2}}|. \tag{2.2}$$

Proof. Consider functions



Thus,

$$\sum_{|n|<N}\Big(1-\frac{|n|}{N}\Big)\prod_{j=1}^d \tau_j^-(n\theta_j) < |\Lambda_{\theta,\varepsilon,N}| < 2\sum_{|n|<2N}\Big(1-\frac{|n|}{2N}\Big)\prod_{j=1}^d \tau_j^+(n\theta_j) \tag{2.3}$$

and

$$\sum_{|n|<N}\Big(1-\frac{|n|}{N}\Big)\prod_{j=1}^d \tau_j^-(n\theta_j) = \sum_{k\in\mathbb{Z}^d}\prod_{j=1}^d \widehat{\tau}_j^-(k_j)F_N(k.\theta)$$

$$= \sum_{k\in\mathbb{Z}^d}\prod_{j=1}^d \frac{\sin^2 \pi\varepsilon_j k_j}{\varepsilon_j\pi^2 k_j^2}F_N(k.\theta) \tag{2.4}$$

$$2\sum_{|n|<2N}\Big(1-\frac{|n|}{2N}\Big)\prod_{j=1}^d \tau_j^+(n\theta_j) = 2\sum_{k\in\mathbb{Z}^d}\prod_{j=1}^d \frac{\sin^2 2\pi\varepsilon_j k_j}{\varepsilon_j\pi^2 k_j^2}F_{2N}(k.\theta). \tag{2.5}$$

Clearly, from $k=0$ contribution and positivity,

$$\text{(2.4)} > \prod \varepsilon_j.F_N(0) = \frac{1}{2}\prod \varepsilon_j.N, \tag{2.6}$$

implying (2.1).

Since

$$F_{2N}(x) \leq 4F_{N/2}(x)$$

$$\sin^2 2x = 4\sin^2 x\cos^2 x \leq 4\sin^2 x \leq 16\sin^2\frac{x}{2},$$

it follows that

$$\text{(2.5)} \leq 8^{d+1}\sum_{k\in\mathbb{Z}^d}\prod_{j=1}^d \frac{\sin^2 \frac{\pi}{2}\varepsilon_j k_j}{\varepsilon_j\frac{\pi^2}{2}k_j^2}F_{\frac{N}{2}}(k.\theta) \tag{2.7}$$

$$\leq 8^{d+1}|\Lambda_{\theta,\frac{\varepsilon}{2},\frac{N}{2}}|, \tag{2.8}$$

proving (2.2). □

§3. Regular Values of (ε, N)

Lemma 3.0. *For given* $(\varepsilon, N) \in \mathbb{R}_+^d \times \mathbb{Z}_+$, *there exist*

$$\varepsilon/2 < \varepsilon_1 < \varepsilon \tag{3.1}$$

$$N/2 < N_1 < N \tag{3.2}$$

such that for $0 < \kappa < 1$,

$$1 - \kappa < \frac{|\Lambda_{\theta,\varepsilon_2,N_2}|}{|\Lambda_{\theta,\varepsilon_1,N_1}|} < 1 + \kappa \tag{3.3}$$

if

$$|\varepsilon_1 - \varepsilon_2| < \frac{1}{100}\frac{\kappa}{d}\varepsilon_1 \quad \textit{in } \mathbb{R}_+^d \tag{3.4}$$

and

$$|N_1 - N_2| < \frac{1}{100}\frac{\kappa}{d}N_1. \tag{3.5}$$

Proof. Assume for each $t \in [1/2, 1]$, there is $\kappa = \kappa(t) \lesssim 1$ such that

$$|\Lambda_{\theta,(1-\frac{1}{100}\frac{\kappa}{d})t\varepsilon,(1-\frac{1}{100}\frac{\kappa}{d})tN}| < (1+\kappa)^{-1}|\Lambda_{\theta,(1+\frac{1}{100}\frac{\kappa}{d})t\varepsilon,(1+\frac{1}{100}\frac{\kappa}{d})tN}|. \tag{3.6}$$

From a standard covering argument of $[1/2, 1]$ by a collection of intervals, we deduce that

$$\frac{|\Lambda_{\theta,\frac{\varepsilon}{4},\frac{N}{4}}|}{|\Lambda_{\theta,2\varepsilon,2N}|} \le \prod_\alpha \frac{|\Lambda_{\theta,(1-\frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha\varepsilon,(1-\frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha N}|}{|\Lambda_{\theta,(1+\frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha\varepsilon,(1+\frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha N}|} \le \prod_\alpha (1+\kappa_\alpha)^{-1}, \tag{3.7}$$

where the intervals $[(1 - \frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha, (1 + \frac{1}{100}\frac{\kappa_\alpha}{d})t_\alpha]$ are disjoint of total measure

$$\frac{1}{50d}\sum \kappa_\alpha t_\alpha > \frac{1}{4}. \tag{3.8}$$

Hence

$$\sum \kappa_\alpha > 12d$$

and

$$\prod(1+\kappa_\alpha) > e^{2/3\sum\kappa_\alpha} > e^{8d}. \tag{3.9}$$

On the other hand, (2.2) implies that

$$\frac{|\Lambda_{\theta,\frac{\varepsilon}{4},\frac{N}{4}}|}{|\Lambda_{\theta,2\varepsilon,2N}|} > 8^{-3(d+1)}. \tag{3.10}$$

Thus from (3.7), (3.9), (3.10), we have

$$8^{-3(d+1)} < e^{-8d}, \tag{3.11}$$

a contradiction.

Let $t_1 \in [1/2, 1]$ be such that for all $0 \leq \kappa \leq 1$,

$$(1+\kappa)|\Lambda_{\theta,(1-\frac{1}{100}\frac{\kappa}{d})t_1\varepsilon,(1+\frac{1}{100}\frac{\kappa}{d})t_1 N}| \geq |\Lambda_{\theta,(1+\frac{1}{100}\frac{\kappa}{d})t_1\varepsilon,(1+\frac{1}{100}\frac{\kappa}{d})t_1 N}| \tag{3.12}$$

and take

$$\varepsilon_1 = t_1\varepsilon, \quad N_1 = t_1 N. \tag{3.13}$$

If (3.4), (3.5) hold, then

$$\Lambda_{\theta,(1-\frac{\kappa}{100d})\varepsilon_1,(1-\frac{\kappa}{100d})N_1} \subset \Lambda_{\theta,\varepsilon_2,N_2} \subset \Lambda_{\theta,(1+\frac{\kappa}{100d})\varepsilon_1,(1+\frac{\kappa}{100d})N_1}; \tag{3.14}$$

and by (3.12),

$$\frac{1}{1+\kappa} < \frac{|\Lambda_{\theta,\varepsilon_2,N_2}|}{|\Lambda_{\theta,\varepsilon_1,N_1}|} < 1+\kappa. \tag{3.15}$$

This proves the lemma. □

Definition. We call (ε_1, N_1) satisfying Lemma 3.0 regular.

Lemma 3.16. *Let* $\lambda = \lambda_{\theta,\varepsilon,N}$ *with* (ε, N) *regular and* $\lambda' = \lambda_{\theta,\frac{\kappa}{100d}\varepsilon,\frac{\kappa}{100d}N}$. *Then*

$$\|\lambda * \lambda' - \lambda\|_1 \equiv \|\lambda * \lambda' - \lambda\|_{\ell^1(\mathbb{Z})} < 2\kappa. \tag{3.17}$$

Proof. Write

$$(\lambda * \lambda')(n) = \sum_m \lambda'(m)\lambda(n-m).$$

If $(\lambda * \lambda')(n) \neq 0$, then there is m

$$|m| < \frac{\kappa}{100d}N, |n-m| < N \tag{3.18}$$

such that

$$\|m\theta_j\| < \frac{\kappa}{100d}\varepsilon_j \tag{3.19}$$

$$\|(n-m)\theta_j\| < \varepsilon_j. \tag{3.20}$$

Hence, from (3.18)–(3.20),

$$|n| < \left(1 + \frac{\kappa}{100d}\right)N \tag{3.21}$$

$$\|n\theta_j\| < \left(1 + \frac{\kappa}{100d}\right)\varepsilon_j \tag{3.22}$$

and

$$n \in \Lambda_{\theta,(1+\frac{\kappa}{100d})\varepsilon,(1+\frac{\kappa}{100d})N}. \tag{3.23}$$

Similarly, one sees that if

$$n \in \Lambda_{\theta,(1-\frac{\kappa}{100d})\varepsilon,(1-\frac{\kappa}{100d})N}, \tag{3.24}$$

then

$$(\lambda * \lambda')(n) = \frac{1}{|\Lambda|} = \lambda(n). \tag{3.25}$$

From the preceding,

$$\|\lambda * \lambda' - \lambda\|_1 = \|(\lambda * \lambda') - \lambda\|_{\ell^1(\Lambda_{\theta,(1+\frac{\kappa}{100d})\varepsilon,(1+\frac{\kappa}{100d})N} \setminus \Lambda_{\theta,(1-\frac{\kappa}{100d})\varepsilon,(1-\frac{\kappa}{100d})N})} \tag{3.26}$$

$$\leq \frac{1}{|\Lambda|}\left[|\Lambda_{\theta,(1+\frac{\kappa}{100d})\varepsilon,(1+\frac{\kappa}{100d})N}| - |\lambda_{\theta,(1-\frac{\kappa}{100d})\varepsilon,(1-\frac{\kappa}{100d})N}|\right] \tag{3.27}$$

$$< 2\kappa, \tag{3.28}$$

using Lemma (3.0).

This proves (3.17). □

Lemma 3.29. *Under the assumption of Lemma 3.16, we also have*

$$\|(\lambda * \lambda') - \lambda\|_2 < 2\sqrt{\kappa}\|\lambda\|_2. \tag{3.30}$$

Proof. By (3.17) and the definition of λ, i.e., (1.2),

$$\begin{aligned}\|(\lambda * \lambda') - \lambda\|_2 &\leq \|(\lambda * \lambda') - \lambda\|_1^{1/2}\|(\lambda * \lambda') - \lambda\|_\infty^{1/2} \leq \sqrt{2\kappa}(2\|\lambda\|_\infty)^{1/2} \\ &= 2\sqrt{\kappa}|\Lambda|^{-1/2} \\ &= 2\sqrt{\kappa}\|\lambda\|_2.\end{aligned}$$

□

§4. Estimation of exponential sum

Let $\theta \in \mathbb{T}^d$, $\lambda = \lambda_{\theta,\varepsilon,N}$ with (ε, N) regular.

Lemma 4.0. *Assume* $x \in \mathbb{T}$ *and*

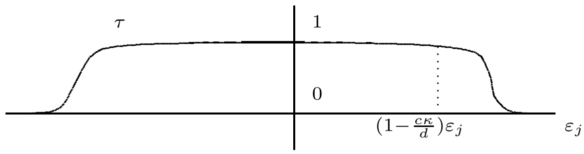
$$\Big|\sum \lambda_n e^{inx}\Big| > \kappa. \tag{4.1}$$

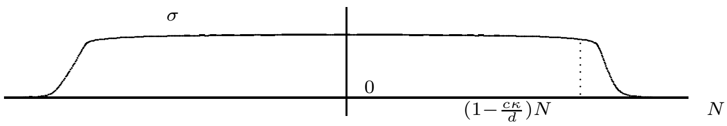
Then, there is $k \in \mathbb{Z}^d$ *such that*

$$|k_j| < Cd\kappa^{-2}\Big(\sum \log\frac{1}{\varepsilon_j}\Big)^2\frac{1}{\varepsilon_j} \tag{4.2}$$

$$\|x - k.\theta\| < Cd^2\kappa^{-2}\Big(\sum \log\frac{1}{\varepsilon_j}\Big)^2\frac{1}{N}. \tag{4.3}$$

Proof. Consider the functions τ_j and σ defined by the graphs below





Here c is an appropriately chosen constant and Fourier transforms $\widehat{\tau}_j, \widehat{\sigma}$ satisfy decay estimates

$$|\widehat{\tau}_j(k)| < 2\varepsilon_j \exp\Big(-\Big(\frac{\kappa\varepsilon_j}{Cd}|k|\Big)^{1/2}\Big) \tag{4.4}$$

$$|\widehat{\sigma}(\lambda)| < 2N \exp\Big(-\Big(\frac{\kappa N}{Cd}|\lambda|\Big)^{1/2}\Big). \tag{4.5}$$

Thus

$$\Big|\sum \sigma_n e^{2\pi inx}\Big| < CN \exp\Big(-\Big(\frac{\kappa N}{Cd}\|x\|\Big)^{1/2}\Big). \tag{4.6}$$

Clearly, from the definition of τ_j and σ, we get

$$\Big|\sum \lambda_n e^{2\pi inx} - \frac{1}{|\Lambda|}\sum \sigma_n \prod_{j=1}^{d} \tau_j(n\theta_j)e^{2\pi inx}\Big| < \frac{1}{|\Lambda|}\big(|\Lambda_{\theta,\varepsilon,N}| - |\Lambda_{\theta,(1-\frac{c\kappa}{d})\varepsilon,\,(1-\frac{c\kappa}{d})N}|\big) < \frac{\kappa}{10} \tag{4.7}$$

for an appropriate choice of c (cf. §3).

Thus, if (4.1),

$$\Big|\sum \sigma_n \prod_{j=1}^{d} \tau_j(n\theta_j)e^{2\pi inx}\Big| > \frac{\kappa}{2}|\Lambda| > \frac{\kappa}{2}\varepsilon_1\cdots\varepsilon_d N, \tag{4.8}$$

by (2.1). Hence

$$\sum_{k\in\mathbb{Z}^d}\prod |\widehat{\tau}_j(k_j)|\,\Big|\sum_n \sigma_n e^{2\pi in(x+k.\theta)}\Big| > \frac{\kappa}{2}\varepsilon_1\cdots\varepsilon_d N; \tag{4.9}$$

and from (4.4), (4.6),

$$\sum_{k\in\mathbb{Z}^d} \exp -\Big[\Big(\frac{\kappa}{Cd}\Big)^{1/2}\sum_{j=1}^{d}(\varepsilon_j|k_j|)^{1/2}+\Big(\frac{\kappa N}{Cd}\Big)^{1/2}\|x+k.\theta\|^{1/2}\Big] > c^d\kappa. \tag{4.10}$$

One has

$$\sum_{k\in\mathbb{Z}} \exp\Big[-\Big(\frac{\kappa\varepsilon|k|}{Cd}\Big)^{1/2}\Big] < \frac{Cd}{\kappa\varepsilon} \tag{4.11}$$

$$\sum_{|k|>k_0} \exp\Big[-\Big(\frac{\kappa\varepsilon|k|}{Cd}\Big)^{1/2}\Big] < \frac{Cd}{\kappa\varepsilon}\exp\Big[-\frac{1}{2}\Big(\frac{\kappa\varepsilon k_0}{Cd}\Big)^{1/2}\Big]. \tag{4.12}$$

Split the sum in (4.10) as

$$\sum_{|k_j|<K_j} + \sum_{\max\frac{|k_j|}{K_j}>1} = (I)+(II). \tag{4.13}$$

Then, by (4.11),

$$(I) < (\varepsilon_1\cdots\varepsilon_d)^{-1}\Big(\frac{Cd}{\kappa}\Big)^d \max_{|k_j|<K_j}\exp\Big(-\Big[\frac{\kappa N}{Cd}\|x+k\theta\|\Big]^{1/2}\Big); \tag{4.14}$$

and by (4.12),

$$(II) < (\varepsilon_1\cdots\varepsilon_d)^{-1}\Big(\frac{Cd}{\kappa}\Big)^d \sum_j \exp\Big[-\frac{1}{2}\Big(\frac{\kappa\varepsilon_j K_j}{Cd}\Big)^{1/2}\Big]. \tag{4.15}$$

Take

$$K_j > \frac{Cd^2}{\kappa^2\varepsilon_j}\Big[\sum_j \log\frac{1}{\varepsilon_j}\Big]^2 > \frac{Cd}{\kappa\varepsilon_j}\Big[\sum_j\log\frac{Cd}{\kappa\varepsilon_j}\Big]^2 \tag{4.16}$$

to insure that

$$(II) < \frac{1}{2}c^d\kappa. \tag{4.17}$$

Hence, by (4.10), (4.13), (4.14), (4.17), we get for some $k\in\mathbb{Z}^d$,

$$|k_j| < K_j \quad (1\le j\le d), \tag{4.18}$$

that

$$\exp -\Big[\frac{\kappa N}{Cd}\|x+k\theta\|\Big]^{1/2} > \frac{1}{2}\Big(\frac{\kappa}{Cd}\Big)^d c^d\kappa\varepsilon_1\cdots\varepsilon_d \tag{4.19}$$

$$\|x+k\theta\| < \frac{Cd}{\kappa N}\Big[\sum_j\log\frac{Cd}{\kappa\varepsilon_j}\Big]^2 < \frac{Cd^2}{\kappa^2 N}\Big(\sum_j\log\frac{1}{\varepsilon_j}\Big)^2. \tag{4.20}$$

From (4.18), (4.16), (4.20), the conclusion (4.2), (4.3) in Lemma 4.0 clearly follows. □

§5. Density

Let $A \subset \{1, \dots, N\}$ satisfy

$$|A| > \delta N. \tag{5.1}$$

For λ a probability measure on $\mathbb{Z}$, define

$$\lambda(A) = \sum_{n \in A} \lambda_n. \tag{5.2}$$

Starting from $\lambda_0 = \frac{1}{2N+1} \mathbb{1}_{\{-N,\dots,N\}}$ and assuming A does not contain a nontrivial triple in progression, we construct a sequence of probability measures λ of the form $\lambda = \lambda_{\theta,\varepsilon,M}$ for varying $d, \theta \in \mathbb{T}^d, \varepsilon \in \mathbb{R}_+^d$ and M such that at each step, $\lambda(A')$ increases suitably for some translate A' of A.

We agree, when introducing measures of the form $\lambda_{\theta,\varepsilon,M}$, always to assume (ε, M) regular.

The main issue in the argument is then how $d, \theta, \varepsilon, M$ evolve along the iteration.

Assume for some translate A' of A

$$\lambda(A') = \delta_1 \geq \delta, \tag{5.3}$$

where $\lambda = \lambda_{\theta,\varepsilon,M}$.

Fix $\kappa > 0$, to be specified, and define

$$\lambda' = \lambda_{\theta, \frac{c\kappa}{d}\varepsilon, \frac{c\kappa}{d} M} \tag{5.4}$$

$$\lambda'' = \lambda_{\theta, (\frac{c\kappa}{d})^2 \varepsilon, (\frac{c\kappa}{d})^2 M}. \tag{5.5}$$

Let λ''' denote the measure

$$\begin{aligned} \lambda'''_n &= \lambda''_{\frac{n}{2}} \quad \text{if } n \in 2\mathbb{Z} \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{5.6}$$

Thus

$$\lambda''' = \lambda_{\tilde{\theta}, (\frac{c\kappa}{d})^2 \varepsilon, 2(\frac{c\kappa}{d})^2 M}, \tag{5.7}$$

where

$$\tilde{\theta} = \frac{\theta}{2} \cup \left\{\frac{1}{2}\right\}. \tag{5.8}$$

Observe that

$$\Lambda_{\tilde{\theta}, \varepsilon', M'} \subset \Lambda_{\theta, 2\varepsilon', M'}. \tag{5.9}$$

According to Lemma 3.16 and the preceding regularity assumption, it follows that

$$\|\lambda - (\lambda * \lambda')\|_1 < \kappa \tag{5.10}$$

$$\|\lambda' - (\lambda' * \lambda'')\|_1 < \kappa \tag{5.11}$$

$$\|\lambda' - (\lambda' * \lambda''')\|_1 < \kappa \tag{5.12}$$

(for an appropriate choice of constants c in (5.4), (5.5)).

Assume that for each $m \in \mathbb{Z}$,

$$|\lambda'(A'+m) - \lambda(A')| > 10\kappa \quad \text{or} \quad |\lambda''(A'+m) - \lambda(A')| > 10\kappa. \tag{5.13}$$

Then, clearly, for either $\lambda^1 = \lambda'$ or $\lambda^1 = \lambda''$,

$$\sum \lambda_m |\lambda^1(A'-m) - \lambda(A')| > 5\kappa. \tag{5.14}$$

Since, by (5.10), (5.11), also

$$\left| \sum \lambda_m [\lambda^1(A'-m) - \lambda(A')] \right| = |(\lambda * \lambda^1)(A') - \lambda(A')| < \|(\lambda * \lambda^1) - \lambda\|_1 < 3\kappa, \tag{5.15}$$

it follows that for some m,

$$\lambda^1(A'+m) > \lambda(A') + \kappa. \tag{5.16}$$

Hence, there is either some translate $A'' = A' + m$ of A satisfying

$$|\lambda'(A'') - \lambda(A')| < 10\kappa; \quad |\lambda''(A'') - \lambda(A')| < 10\kappa \tag{5.17}$$

or, for some translate $A'' = A' + m$, there is a density increment

$$\lambda'(A'') > \lambda(A') + \kappa \quad \text{or} \quad \lambda''(A'') > \lambda(A') + \kappa. \tag{5.18}$$

In the sequel, we assume

$$\delta_1^2 \ll \kappa \ll \delta_1. \tag{5.19}$$

§6. Comparison of the Integrals

Assume (5.17) for some translate A'' of A. Following the circle method, consider the sums

$$S' = \sum \lambda'_n \, e^{2\pi inx} \tag{6.1}$$

$$S'_A = \sum_{n\in A''} \lambda'_n \, e^{2\pi inx} \tag{6.2}$$

$$S'' = \sum \lambda''_n e^{2\pi inx} \tag{6.3}$$

$$S''_A = \sum_{n\in A''} \lambda''_n \, e^{2\pi inx} \tag{6.4}$$

$$S''' = \sum \lambda'''_n \, e^{2\pi inx}. \tag{6.4$'$}$$

Since A (and hence A'') does not contain a nontrivial triple in progression,

$$I_1 \equiv \int_{\mathbb{T}} S'_A(x)^2 S''_A(-2x)dx = \sum_{n\in A''} (\lambda'_n)^2 \lambda''_n. \tag{6.5}$$

On the other hand,

$$\begin{aligned} I_2 \equiv &\int_{\mathbb{T}} [\lambda'(A'')S'(x)]^2 \ [\lambda''(A'')S''(-2x)]dx \\ =&\lambda'(A'')^2\lambda''(A'') \sum_{n_1+n_2=2m} \lambda'_{n_1}\lambda'_{n_2}\lambda''_m \\ =&\lambda'(A'')^2\lambda''(A'') \sum_{n,m} \lambda'_n \, \lambda'_{n-2m} \, \lambda''_m. \end{aligned} \tag{6.6}$$

By construction of $\lambda', \lambda'', \lambda'''$, cf. (5.6), (5.12), we have

$$\sum_n \Big|\lambda'_n - \Big(\sum_m \lambda'_{n-2m} \, \lambda''_m\Big)\Big| < \|\lambda' - (\lambda' * \lambda''')\|_1 < \kappa$$

$$\Big(\sum_n \Big|\lambda'_n - \Big(\sum_m \lambda'_{n-2m} \, \lambda''_m\Big)\Big|^2\Big)^{1/2} < \kappa^{1/2}\|\lambda'\|_\infty^{1/2} = \kappa^{1/2}\|\lambda'\|_2. \tag{6.7}$$

Hence, from (5.17), (5.19), (6.7), it follows that

$$(6.6) > (\delta_1 - 10\kappa)^3(1-\kappa^{1/2})\|\lambda'\|_2^2 \tag{6.8}$$

$$\overset{(5.19)}{>} \frac{1}{2}\, \delta_1^3\|\lambda'\|_2^2. \tag{6.9}$$

We assume throughout the construction of the measures $\lambda = \lambda_{\theta,\varepsilon,M}$, $\theta \in \mathbb{T}^d$, that the condition

$$\log M \gg \sum \log \frac{1}{\varepsilon_j} + d\log\frac{d}{\delta} \tag{6.10}$$

is satisfied.

Thus

$$(6.11)\qquad (6.5)\ < \frac{1}{|\Lambda''|}\sum(\lambda_n')^2 < \frac{1}{M(\frac{c\kappa}{d})^{2(d+1)}\varepsilon_1\cdots\varepsilon_d}\,\|\lambda'\|_2^2 < M^{-1/2}\|\lambda'\|_2^2;$$

and from (6.6), (6.9), (6.11),

$$(6.12)\qquad |I_1 - I_2| > \frac{1}{3}\,\delta_1^3\|\lambda'\|_2^2.$$

Estimate

$$(6.13)\qquad |I_1 - I_2| \le \int_{\mathbb{T}} |S_A'(x)|^2.|S_A'' - \lambda''(A'')S''|(-2x)$$

$$(6.14)\qquad + \int_{\mathbb{T}} |S_A'(x)^2 - [\lambda'(A'')S'(x)]^2|\,\lambda''(A'')|S''(-2x)|.$$

§7. Density Increment (I)

Assume first that

$$(7.1)\qquad (6.14) > \frac{1}{6}\delta_1^3\|\lambda'\|_2^2,$$

Since

$$(7.2)\quad \|(S_A')^2 - [\lambda'(A'')S']^2\|_1 \le \|S_A'\|_2^2 + \lambda'(A'')^2\|\lambda'\|_2^2 < 2\lambda'(A'')\|\lambda'\|_2^2 < 3\delta_1\|\lambda'\|_2^2,$$

(7.1) clearly implies that

$$(7.3)\qquad \int_{\mathcal{F}} \big|(S_A')^2 - [\lambda'(A'')S']^2\big| > \frac{\delta_1^2}{10}\|\lambda'\|_2^2,$$

where

$$(7.4)\qquad \mathcal{F} = \{x \in \mathbb{T} : |S''(-2x)| > 10^{-3}\delta_1\}.$$

Estimating the left side of (7.3) as

$$\|S_A' - \lambda'(A'')S'\big|_{\mathcal{F}}\|_2^2 + \lambda'(A'')\|\lambda'\|_2\,\|S_A' - \lambda'(A'')S'\big|_{\mathcal{F}}\|_2,$$

we see that

$$(7.5)\qquad \|S_A' - \lambda'(A'')S'\big|_{\mathcal{F}}\|_2 > \frac{\delta_1}{10}\|\lambda'\|_2.$$

In order to specify $\mathcal{F}$, apply Lemma 4.0 with λ replaced by λ''' given by (5.7). Thus if $x \in \mathcal{F}$, there exists $k \in \mathbb{Z}^d$ such that

$$(7.6)\qquad |k_j| < C\frac{d^5}{\delta_1^7}\Big(\sum \log\frac{1}{\varepsilon_{j'}}\Big)^2\frac{1}{\varepsilon_j} \quad (1 \le j \le d)$$

and

$$\|x - k\tilde{\theta}\| < C\frac{d^5}{\delta_1^7}\Big(\sum \log \frac{1}{\varepsilon_{j'}}\Big)^2 \frac{1}{M}, \tag{7.7}$$

where $\tilde{\theta}$ is given by (5.8).

Thus, if we let

$$\overset{\approx}{\Lambda} = \Lambda_{\tilde{\theta}, \overset{\approx}{\varepsilon}, \overset{\approx}{M}} \tag{7.8}$$

with

$$\overset{\approx}{\varepsilon} = c\frac{\delta_1^9}{d^6}\Big(\sum \log \frac{1}{\varepsilon_j}\Big)^{-2} \varepsilon \tag{7.9}$$

$$\overset{\approx}{M} = c\frac{\delta_1^9}{d^6}\Big(\sum \log \frac{1}{\varepsilon_j}\Big)^{-2} M, \tag{7.10}$$

it follows from (7.6), (7.7) that for $x \in \mathcal{F}$ and $n \in \overset{\approx}{\Lambda}$,

$$\begin{aligned} \|nx\| &\le \sum_j |k_j| \, \|n\tilde{\theta}_j\| + |n| \text{ (7.7)} \\ &< \frac{d^5}{\delta_1^7}\Big(\sum \log \frac{1}{\varepsilon_j}\Big)^2 \sum \overset{\approx}{\varepsilon}_j / \varepsilon_j + \overset{\approx}{M}. \text{(7.7)} \\ &< 10^{-3}\delta_1^2. \end{aligned} \tag{7.11}$$

Recall (5.4), (5.19)

$$\lambda' = \lambda_{\theta, c\frac{x_1}{d}\varepsilon, c\frac{x_1}{d}M}; \tag{7.12}$$

then from (7.9), (7.10), (3.16), the multiplier $\overset{\approx}{\lambda}$ associated with $\overset{\approx}{\Lambda}$ also satisfies

$$\|(\lambda' * \overset{\approx}{\lambda}) - \lambda'\|_1 < 10^{-6}\delta_1^5 \tag{7.13}$$

Hence

$$\|(\lambda' * \overset{\approx}{\lambda}) - \lambda'\|_2 < 10^{-3}\delta_1^{5/2}\|\lambda'\|_2. \tag{7.14}$$

Write

$$\begin{aligned} S'_A(x) &= \sum_{n \in A''} \lambda'_n e^{2\pi i n x} \\ &= \sum_{n \in A''} (\lambda' * \overset{\approx}{\lambda})_n e^{2\pi i n x} && (7.15) \\ &\quad + \sum_{n \in A''} \big(\lambda' - (\lambda' * \overset{\approx}{\lambda})\big)_n e^{2\pi i n x}. && (7.16) \end{aligned}$$

From (7.14),

$$\|(7.16)\|_2 \le \|\lambda' - (\lambda' * \overset{\approx}{\lambda})\|_2 < 10^{-3}\delta_1^{5/2}\|\lambda'\|_2. \tag{7.17}$$

Write

$$(7.15) = \sum_{n\in A'',m} \lambda'_m \overset{\approx}{\lambda}_{n-m}\, e^{2\pi i n x}$$

$$= \sum_m \lambda'_m\, e^{2\pi i m x}\, \overset{\approx}{\lambda}(A''-m) \tag{7.18}$$

$$+ \sum_{n\in A'',m} \lambda'_m \overset{\approx}{\lambda}_{n-m}(e^{2\pi i n x} - e^{2\pi i m x}). \tag{7.19}$$

By (7.11), one has for $x \in \mathcal{F}$

$$|(7.19)| = \Big| \sum_{n\in A'',m} \lambda'_{n-m} \overset{\approx}{\lambda}_m (e^{2\pi i n x} - e^{2\pi i (n-m)x})\Big|$$

$$= \Big| \sum_m \overset{\approx}{\lambda}_m (e^{2\pi i m x} - 1)\Big[\sum_{n\in A''} \lambda'_{n-m}\, e^{2\pi i (n-m)x}\Big]\Big|$$

$$\le 10^{-3}\delta_1^2 \Big[\sum_m \overset{\approx}{\lambda}_m\Big] \Big|\sum_{k\in A''-m} \lambda'_k\, e^{2\pi i k x}\Big|; \tag{7.20}$$

hence

$$\big\|(7.19)\big|_{\mathcal{F}}\big\|_2 \le 10^{-3}\delta_1^2\, \|\lambda'\|_2. \tag{7.21}$$

Thus, from (7.17), (7.21),

$$\|[S'_A - \lambda'(A'')S']\big|_{\mathcal{F}}\|_2 < \Big\|\sum_m \lambda'_m e^{2\pi i m x}\, \overset{\approx}{\lambda}(A''-m) - \lambda'(A'')S'\Big\|_2 + \frac{1}{500}\delta_1^2\|\lambda'\|_2$$

$$= \Big(\sum_m (\lambda'_m)^2[\lambda'(A'') - \overset{\approx}{\lambda}(A''-m)]^2\Big)^{1/2} + \frac{\delta_1^2}{500}\|\lambda'\|_2. \tag{7.22}$$

Consequently, (7.5), (7.22) give

$$\Big(\sum_m (\lambda'_m)^2[\lambda'(A'') - \overset{\approx}{\lambda}(A''-m)]^2\Big)^{1/2} > \frac{\delta_1}{200}\|\lambda'\|_2$$

$$\sum \lambda'_m[\lambda'(A'') - \overset{\approx}{\lambda}(A''-m)]^2 > \frac{\delta_1^2}{4.10^4} \tag{7.23}$$

and

$$[\lambda'(A'') + \max_m \overset{\approx}{\lambda}(A''-m)]\Big[\sum \lambda'_m|\lambda'(A'') - \overset{\approx}{\lambda}(A''-m)|\Big] > \frac{\delta_1^2}{4.10^4}. \tag{7.24}$$

From (7.24), either for some m

$$\tilde{\tilde{\lambda}}(A'' - m) > \frac{4}{3}\,\delta_1 \tag{7.25}$$

or

$$\sum \lambda'_m |\lambda'(A'') - \tilde{\tilde{\lambda}}(A'' - m)| > \delta_1/10^5. \tag{7.26}$$

Since again

$$\Big|\sum_m \lambda'_m [\lambda'(A'') - \tilde{\tilde{\lambda}}(A'' - m)]\Big| = |\lambda'(A'') - (\lambda' * \tilde{\tilde{\lambda}})(A'')| \overset{(7.13)}{<} 10^{-6}\delta_1^5 \tag{7.27}$$

(7.26), (7.27) imply that

$$\tilde{\tilde{\lambda}}(A'' - m) - \lambda'(A'') > \frac{1}{2}\Big(\frac{\delta_1}{10^5} - \frac{\delta_1^5}{10^6}\Big) > \frac{\delta_1}{10^6}$$

$$\tilde{\tilde{\lambda}}(A'' - m) \overset{(5.17)}{>} \lambda(A') - 10\kappa + 10^{-6}\delta_1 > \delta_1 + 10^{-7}\delta_1 \tag{7.28}$$

for some m. Thus (7.25), (7.28) again give the increment

$$\tilde{\tilde{\lambda}}(\tilde{\tilde{A}}) > \delta_1 + 10^{-7}\delta_1 \tag{7.29}$$

for some translate of A

$$\tilde{\tilde{A}} = A'' - m. \tag{7.30}$$

§8. Density Increment (II)

Assume next

$$(6.13) > \frac{1}{6}\delta_1^3 \|\lambda'\|_2^2. \tag{8.1}$$

Since $\|S'_A\|_2 = \lambda'(A'')^{1/2}\|\lambda'\|_2 \sim \delta_1^{1/2}\|\lambda'\|_2$, (8.1) clearly implies

$$\int_{[-2x\in\mathcal{F}]} |S'_A(x)|^2 |S''_A - \lambda''(A'')S''|(-2x) > \frac{1}{10}\delta_1^3\|\lambda'\|_2^2, \tag{8.2}$$

where

$$\mathcal{F} = \{x \in \mathbb{T} : |S''_A - \lambda''(A'')S''|(x) > \frac{1}{10}\delta_1^2\}. \tag{8.3}$$

Distinguishing level sets, we may further specify some

$$\delta_1/10 < \tau < 2 \tag{8.4}$$

so that

$$\int_{[-2x\in\mathcal{F}_0]} |S'_A(x)|^2 > \frac{\delta_1^2}{\tau\log\frac{1}{\delta_1}}\|\lambda'\|_2^2, \tag{8.5}$$

where

$$\mathcal{F}_0 = \{x\in\mathbb{T} : |S''_A - \lambda''(A'')S''|(x) \sim \tau\delta_1\}. \tag{8.6}$$

In particular, $\mathcal{F}_0 \neq \emptyset$. This fact allows us to perform a first density increment (that will be advantageous for 'large' τ). Take $x_0 \in \mathbb{T}$ with

$$|S''_A(x_0) - \lambda''(A'')S''(x_0)| > \tau\delta_1. \tag{8.7}$$

Recalling (5.5) where $\theta\in\mathbb{T}^d$, replace d by $d+1$ and θ by $\tilde\theta = \theta\cup\{x_0\}\in\mathbb{T}^{d+1}$. Take

$$\mathbb{R}_+^{d+1} \ni \tilde\varepsilon = c\frac{\delta_1^2}{d}\Big(\Big(\frac{c\kappa}{d}\Big)^2\varepsilon, 1\Big) \tag{8.8}$$

$$\tilde M = c\frac{\delta_1^2}{d}\Big(\frac{c\kappa}{d}\Big)^2 M$$

and let

$$\tilde\lambda = \lambda_{\tilde\theta,\tilde\varepsilon,\tilde M}. \tag{8.9}$$

Then

$$S''_A(x_0) = \sum_{n\in A''}\lambda''_n e^{2\pi i n x_0} = \sum_{n\in A'',m}\lambda''_m\tilde\lambda_{n-m}e^{2\pi i n x_0} + 0(\|\lambda'' - (\lambda''*\tilde\lambda)\|_1) \tag{8.10}$$

$$\overset{(3.16)}{=} \sum \lambda''_m e^{2\pi i m x_0}\tilde\lambda(A''-m) + 0\Big(\sum\tilde\lambda_n|e^{2\pi i n x_0} - 1| + c\delta_1^2\Big) \tag{8.11}$$

$$< \sum\lambda''_m\tilde\lambda(A''-m)e^{2\pi i m x_0} + c\delta_1^2, \tag{8.12}$$

since $\|nx_0\| < \frac{c}{d}\delta_1^2$ for $n\in\tilde\Lambda$.

Thus from (8.4), (8.7), (8.12),

$$\Big|\sum_m \lambda''_m[\tilde\lambda(A''-m) - \lambda''(A'')e^{2\pi i m x_0}\Big| > \frac{1}{2}\delta_1\tau \tag{8.13}$$

$$\sum\lambda''_m|\tilde\lambda(A''-m) - \lambda''(A'')| > \frac{1}{2}\delta_1\tau. \tag{8.14}$$

Again,

$$\Big|\sum\lambda''_m[\tilde\lambda(A''-m) - \lambda''(A'')]\Big| = |(\lambda''*\tilde\lambda)(A'') - \lambda''(A'')| < c\delta_1^2. \tag{8.15}$$

From (8.14), (8.15), we conclude that for some translate $\tilde{A} = A'' - m$ of A,

$$\tilde{\lambda}(\tilde{A}) > \lambda''(A'') + \frac{1}{2}\Big(\frac{1}{2}\delta_1\tau - c\delta_1^2\Big) > \lambda''(A'') + \frac{1}{5}\tau\delta_1. \tag{8.16}$$

Assume

$$\tau\delta_1 > 10\kappa. \tag{8.17}$$

Recalling (5.18), we obtain the density increment

$$\tilde{\lambda}(\tilde{A}) > \delta_1 - \kappa + \frac{1}{5}\tau\delta_1 > \Big(1 + \tau/10\Big)\delta_1 > \delta_1 + \kappa. \tag{8.18}$$

§9. Density Increment (III)

Our next aim is to exploit (8.5) to produce a density increment (to be used when τ is 'small', more precisely, $\tau\delta_1 < 10\kappa$).

Depending on the structure of $\mathcal{F}_0$, we distinguish two cases. Recall (8.5)

$$\|S'_A|_{\mathcal{G}}\|_2 > \frac{\delta_1}{\sqrt{\tau \log \frac{1}{\delta_1}}}\|\lambda'\|_2, \tag{9.1}$$

where $\mathcal{G} = \{x \in \mathbb{T} : -2x \in \mathcal{F}_0\}$.

Let $t_1, \ldots, t_{R-1}$ be a maximal subset of $\mathcal{G}$ subject to the condition that for each $r = 1, \ldots, R-1$

$$\Big\|2t_r + \sum_{j=1}^{d} k_j\theta_j + 2\sum_{s\neq r}\ell_s t_s\Big\| > \frac{1}{\gamma M} \tag{9.2}$$

whenever $k_j \in \mathbb{Z}, |k_j| < K_j$ and $\ell_s \in \{0, 1, -1\}$ with $0 < \gamma < 1$ and K_j to be specified later.

Set

$$t_R = 1/2.$$

Thus if $x \in \mathcal{G}$, then for some $k_j \in \mathbb{Z}, |k_j| < K_j$ and $\ell_r \in \{0, 1, -1\}$, we have

$$\Big\|x + \frac{1}{2}\sum_{j=1}^{d} k_j\theta_j + \sum_{r=1}^{R}\ell_r t_r\Big\| \leq \frac{1}{\gamma M}. \tag{9.3}$$

Let then

$$\tilde{\theta} = \Big(\frac{1}{2}\theta_1, \ldots, \frac{1}{2}\theta_d, t_1, \ldots, t_R\Big) \in \mathbb{T}^{d+R}$$

and

$$\tilde{\varepsilon} \in \mathbb{R}_+^{d+R}, \quad \tilde{M} \in \mathbb{Z}_+$$

to be specified.

For

$$n \in \tilde{\Lambda} = \Lambda_{\tilde{\theta},\tilde{\varepsilon},\tilde{M}}$$

and $x \in \mathcal{G}$, (9.3) implies

$$\|nx\| < \frac{\tilde{M}}{\gamma M} + \sum_{j=1}^{d} K_j \tilde{\varepsilon}_j + \sum_{r=1}^{R} \tilde{\varepsilon}_{d+r} < 10^{-3}\delta_1^4 \tag{9.4}$$

if we assume

$$\tilde{M} < \frac{\gamma}{d^2}\delta_1^6 M \tag{9.5}$$

$$\tilde{\varepsilon}_j < \frac{\delta_1^4}{2dK_j} \text{ for } 1 \le j \le d \tag{9.6}$$

$$\tilde{\varepsilon}_j < \frac{\delta_1^4}{2d} \text{ for } d+1 \le j \le d+R. \tag{9.7}$$

Proceed as in §7.

Recall that

$$\lambda' = \lambda_{\theta, c\frac{\kappa}{d}\varepsilon, c\frac{\kappa}{d}M},$$

and let $\tilde{\lambda}$ be associated to $\tilde{\Lambda}$ introduced above.

Assume, moreover, that

$$\tilde{\varepsilon}_j < c\frac{\delta_1^6}{d^2}\varepsilon_j \text{ for } 1 \le j \le d. \tag{9.8}$$

From (9.5), (9.8),

$$\|\lambda' - \lambda' * \tilde{\lambda}\|_1 < 10^{-3}\delta_1^4 \tag{9.9}$$

and

$$\|\lambda' - \lambda' * \tilde{\lambda}\|_2 < 10^{-3}\delta_1^2 \|\lambda'\|_2. \tag{9.10}$$

Write

$$S'_A(x) = \sum_{n \in A''} \lambda'_n e^{2\pi i n x} = \sum_{n \in A''} (\lambda' * \tilde{\lambda})_n e^{2\pi i n x} + \text{(9.11)},$$

where (9.11) is defined by above equation and by (9.10),

$$\|\text{(9.11)}\|_2 < 10^{-3}\delta_1^2 \|\lambda'\|_2. \tag{9.12}$$

Write

$$\sum_{n \in A''} (\lambda' * \tilde{\lambda})_n e^{2\pi i n x} = \tag{9.13}$$
$$\sum_{n \in A'', m} \lambda'_m \tilde{\lambda}_{n-m} e^{2\pi i m x} + \sum_{n \in A'', m} \lambda'_m \tilde{\lambda}_{n-m} e^{2\pi i m x}[e^{2\pi i (n-m)x} - 1].$$

Invoking (9.4), we see that for $x \in \mathcal{G}$, the second term in (9.13) is at most

$$(9.14)\qquad \Big|\sum_m \tilde\lambda_m [e^{2\pi imx}-1]\Big[\sum_{n\in A''}\lambda'_{n-m}e^{2\pi i(n-m)x}\Big]\Big| \le \\ 10^{-3}\delta_1^4\sum_m\tilde\lambda_m\Big|\sum_{k\in A''-m}\lambda'_k e^{2\pi ikx}\Big|.$$

Thus

$$\sum_{n\in A''}(\lambda'*\tilde\lambda)_n e^{2\pi inx}=\sum_m\tilde\lambda(A''-m)\lambda'_m e^{2\pi imx}+(9.15)$$

with

$$(9.16)\qquad \|(9.15)|_{\mathcal{G}}\|_2<10^{-3}\delta_1^4.\|\lambda'\|_2.$$

From (9.1), (9.12), (9.16), it follows that

$$\Big[\sum_m(\tilde\lambda(A''-m)\lambda'_m)^2\Big]^{1/2}\ge\Big\|\sum_m\tilde\lambda(A''-m)\lambda'_m e^{2\pi imx}\Big|_{\mathcal{G}}\Big\|_2 \\ >\Big(\frac{\delta_1}{\sqrt{\tau\log\frac{1}{\delta_1}}}-10^{-3}\delta_1^2-10^{-3}\delta_1^4\Big)\|\lambda'\|_2.$$

Hence

$$(9.17)\qquad \sum\lambda'_m\tilde\lambda(A''-m)^2>\delta_1^2\Big(\tau\log\frac{1}{\delta_1}\Big)^{-1}.$$

The left side of (9.17) is at most

$$[\max_m\tilde\lambda(A''-m)].(\lambda'*\tilde\lambda)(A'')\underset{(9.9)}{<}\big(\lambda'(A'')+10^{-4}\delta_1^4\big)\max_m\tilde\lambda(A''-m) \\ <2\delta_1\max_m\tilde\lambda(A''-m);$$

and we conclude that for some translate $\tilde A=A''-m$ of A,

$$(9.18)\qquad \tilde\lambda(\tilde A)>\frac{\delta_1}{2\tau\log\frac{1}{\delta_1}}.$$

In the definition of $\tilde\lambda$, from (9.5)–(9.8), $\tilde d=d+R$ and $\tilde M,\tilde\varepsilon$ are subject to the conditions

$$(9.19)\qquad \tilde M<\frac{\gamma}{d^2}\delta_1^6 M$$

and

$$(9.20)\qquad \begin{cases}\tilde\varepsilon_j<c\frac{\delta_1^6}{d^2}\min(\varepsilon_j,\frac{1}{K_j})\text{ for }1\le j\le d\\ \tilde\varepsilon_j<\frac{\delta_1^4}{2d}\text{ for }d<j\le d+R.\end{cases}$$

The parameters γ and K_j $(1 \le j \le d)$ remain to be chosen.

The construction described in this section leading to (9.13) is useful only if R is not too 'large' (to be specified later). If R is large, we follow a different procedure described next.

§10. Density Increment (IV)

This section contains the new analytical ingredients.

Recalling the definition of $\mathcal{F}_0$ and $\mathcal{G}$, denote $\psi_r = -2t_r (1 \le r < R)$. Thus $\psi_r \in \mathcal{F}_0$; and from (9.2),

$$\Big\| \psi_r + \sum_{j=1}^{d} k_j \theta_j + \sum_{s \neq r} \ell_s \psi_s \Big\| > \frac{1}{\gamma M} \tag{10.1}$$

whenever $k_j \in \mathbb{Z}$, $|k_j| < K_j$ and $\ell_s \in \{0, 1, -1\}$.

Recall that

$$S''_A(x) = \sum_{n \in A''} \lambda''_n e^{2\pi i n x}, \tag{10.2}$$

where

$$\lambda'' = \lambda_{\theta, \varepsilon'' M''} \tag{10.3}$$

and

$$\varepsilon'' = \Big(\frac{c\kappa}{d}\Big)^2 \varepsilon \text{ and } M'' = \Big(\frac{c\kappa}{d}\Big)^2 M. \tag{10.3$'$}$$

Returning to (8.6), we have for each $1 \le r < R$, either

$$|S''_A(\psi_r)| > \frac{1}{3} \tau \delta_1 \tag{10.4}$$

or

$$|S''(\psi_r)| > \frac{1}{3} \tau. \tag{10.5}$$

If (10.5), application of Lemma 4.0 to λ'' implies by (4.2), (4.3), (10.3′) that

$$\|\psi_r - k.\theta\| < C \frac{d^5}{\kappa^3} \tau^{-2} \Big(\sum \log \frac{1}{\varepsilon_j}\Big)^2 \frac{1}{M} \tag{10.6}$$

with $k \in \mathbb{Z}^d$ satisfying

$$|k_j| < c \frac{d^4}{\kappa^3} \tau^{-2} \Big(\sum \log \frac{1}{\varepsilon_{j'}}\Big)^2 \frac{1}{\varepsilon_j}. \tag{10.7}$$

In order for (10.6) to contradict (10.1), take in (9.2), (9.20)

$$K_j \sim cd^6\delta_1^{-14}\Big(\sum \log \frac{1}{\varepsilon_{j'}}\Big)^2 \frac{1}{\varepsilon_j} \quad (1 \le j \le d). \tag{10.8}$$

and in (9.19)

$$\gamma = cd^{-6}\delta_1^{10}\Big(\sum \log \frac{1}{\varepsilon_j}\Big)^{-2}. \tag{10.9}$$

Hence

$$|S''_A(\psi_r)| > \frac{1}{3}\tau\delta_1 \text{ for } 1 \le r \le R. \tag{10.10}$$

Next, fix $\mathcal{R} \subset \{1, \dots, R-1\}, |\mathcal{R}| < \delta_1^{-2}$ and let $a_r \in \mathbb{C}, |a_r| \le \frac{1}{4}$ for $r \in \mathcal{R}$. Set

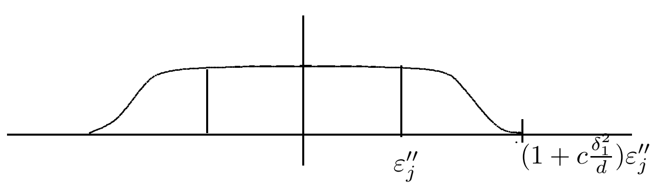
$$0 < \Omega_n = \prod_{r \in \mathcal{R}} (1 + Re\, a_r e^{2\pi i n\psi_r}). \tag{10.11}$$

We first analyze the expression

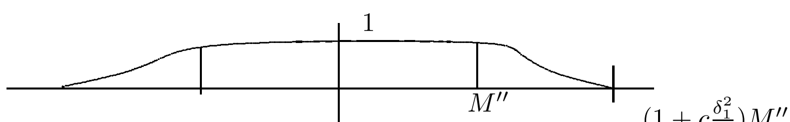
$$\sum_n \lambda''_n \Omega_n, \tag{10.12}$$

again relying on (10.1).

Proceeding as in §4, define for each $j = 1, \dots, d$ the smooth function τ_j



and σ



In particular, recalling (10.3), our construction gives

$$\mathcal{X}_{\Lambda''}(n) \le \sum \sigma_n \prod_{j=1}^{d} \tau_j(n\theta_j) \le \mathcal{X}_{\bar{\Lambda}}(n), \tag{10.13}$$

where

$$\bar{\Lambda} = \Lambda_{\theta,(1+c\frac{\delta_1^2}{d})\varepsilon'',(1+c\frac{\delta_1^2}{d})M''}. \tag{10.14}$$

From regularity and Lemma 3.0,

$$|\bar{\Lambda}| - |\Lambda''| < c\delta_1^2|\Lambda''| \tag{10.15}$$

$$\|\lambda' - \lambda''\|_1 < c\delta_1^2. \tag{10.16}$$

From positivity,

$$\sum_n \lambda_n''\Omega_n \leq \frac{1}{|\Lambda''|}\sum_n \sigma_n \prod_{j=1}^d \tau_j(n\theta_j)\prod_{r\in\mathcal{R}}(1 + Re\, a_r e^{2\pi in\psi_r}), \tag{10.17}$$

where the right side of (10.17) equals

$$\frac{1}{|\Lambda''|}\sum_n \sigma_n \prod_{j=1}^d \tau_j(n\theta_j) \tag{10.18}$$

(10.19)

$$+\frac{1}{|\Lambda''|}O\Big(\sum_{1\leq|s|\leq|\mathcal{R}|} 8^{-s} \sum_{\substack{r_1<\cdots<r_s\in\mathcal{R}\\ \nu_1,\ldots,\nu_s=\pm1}} \Big|\sum_n \sigma_n\prod_{j=1}^d \tau_j(n\theta_j)e\big(n(\nu_1\psi_{r_1}+\cdots+\nu_s\psi_{r_s})\big)\Big|\Big).$$

From (10.13), (10.15),

$$(10.18) \leq \frac{|\bar{\Lambda}|}{|\Lambda''|} < 1 + c\delta_1^2. \tag{10.20}$$

We estimate the inner sum in (10.19) by Fourier expansion of the functions τ_j, as in §4. By construction of τ_j, we may ensure the decay estimates

$$|\widehat{\tau}_j(k)| < 2\varepsilon_j'' \exp\Big(-\Big(c\frac{\delta_1^2}{d}\varepsilon_j''|k|\Big)^{1/2}\Big). \tag{10.21}$$

Also

$$|\widehat{\sigma}(\lambda)| < 2M'' \exp\Big(-\Big(\frac{c\delta_1^2}{d}M''\|\lambda\|\Big)^{1/2}\Big). \tag{10.22}$$

Hence

$$\Big|\sum_n \sigma_n \prod_{j=1}^d \tau_j(n\theta_j)e\big(n(\nu_1\psi_{r_1}+\cdots+\nu_s\psi_{r_s})\big)\Big| \leq$$
$$\sum_{k\in\mathbb{Z}^d}\prod|\tau_j(k_j)|\,\Big|\sum_n \sigma_n e\big(n(k.\theta+\nu_1\psi_{r_1}+\cdots+\nu_s\psi_{r_s})\big)\Big|.$$

Invoking the bounds (10.21), (10.22), we obtain the estimate

$$2^{d+1}\prod_{j=1}^{d}\varepsilon''_j.M''.\sum_{k\in\mathbb{Z}^d}\exp-\Big[\Big(\frac{c\delta_1^2}{d}.\sum(\varepsilon''_j|k_j|)\Big)^{1/2}$$

$$+\Big(\frac{c\delta_1^2}{d}M''\|k\theta+\nu_1\psi_{r_1}+\cdots+\nu_s\psi_{r_s}\|\Big)^{1/2}\Big]. \tag{10.23}$$

Split the summation as

$$\sum_{k\in\mathbb{Z}^d}=\sum_{|k_j|<K_j}+\sum_{\max\frac{|k_j|}{K_j}>1}.$$

Invoking (10.1) and recalling (10.3′) and (10.9), we see that the contribution of the first sum in (10.23) is at most

$$2^{d+1}.\prod_{j=1}^{d}\varepsilon''_j.M''.\exp\Big[-\Big(\frac{c\delta_1^2}{d}\frac{M''}{\gamma M}\Big)^{1/2}\Big]\overset{(2.1)}{<} \tag{10.24}$$

$$2^{d+2}|\Lambda''|\exp\Big[-\Big(\frac{c\delta_1^6}{d^3\gamma}\Big)^{1/2}\Big]<e^{-2\delta_1^{-2}}|\Lambda''|$$

Recalling (10.8), we get the bound

$$|\Lambda''|\sum_{j=1}^{d}\frac{(cd\delta_1^{-2})^d}{\prod\varepsilon''_{j'}}\exp\Big[-\frac{c\delta_1^4}{d^2}(\varepsilon_jK_j)^{1/2}\Big]<e^{-2\delta_1^{-2}}|\Lambda''| \tag{10.25}$$

for the contribution of the second sum.

It follows from (10.24), (10.25) that certainly

$$(10.19)<c\delta_1^2; \tag{10.26}$$

and hence by (10.20), (10.17),

$$\sum\lambda''_n\Omega_n<1+c\delta_1^2. \tag{10.27}$$

At this point, we invoke the following general property (see Appendix).

Proposition $(*)$ **.** *Let* (Ω,μ) *be a probability space and* $(\varphi_j)_{1\le j\le k}$ *real functions on* Ω *satisfying*

(i) $$|\varphi_j|<\frac{1}{2}\quad(1\le j\le k)$$

and

(ii) $$\int\prod_{j=1}^{k}(1+a_j\varphi_j)d\mu<2 \text{ for all } a_j\in[-1,1].$$

Let $A \subset \Omega$, $\mu(A) = \delta$ *and assume that*

(iii) $$\int_A \varphi_j d\mu > \tau\delta \quad (1 \le j \le k).$$

Then

(iv) $$k < C\tau^{-2} \log \frac{1}{\delta},$$

and there is a subset $I \subset \{1, \dots, k\}$ *such that*

(v) $$|I| \sim \frac{\tau k}{\log \frac{1}{\delta}}$$

and

(vi) $$\int_A \prod_{j\in I} (1 + \varphi_j) d\mu > \delta(1 + \frac{\tau}{2}|I|).$$

We apply Proposition $(*)$ to the probability measure λ'' on $\mathbb{Z}$ and the set A''.

If $R < \delta_1^{-2}$, let $\mathcal{R} = \{1, \dots, R-1\}$. Otherwise, take a subset $\mathcal{R} \subset \{1, \dots, R-1\}$ of size δ_1^{-2}. From (10.10), we may take $b_r \in \mathbb{C}$, $|b_r| \le \frac{1}{4}$ such that

(10.28) $$Re\, b_r S''_A(\psi_r) = \sum_{n\in A''} \lambda''_n \, Re(b_r e^{2\pi i n\psi_r}) > \frac{\tau\delta_1}{20}.$$

In Proposition $(*)$, let

$$k = |\mathcal{R}|$$
$$\varphi_r = Re(b_r e^{2\pi i n\psi_r}) \qquad \text{for } r \in \mathcal{R}$$

Then (10.27) implies condition (ii), and (10.28) implies (iii). From (iv), it follows that in fact

(10.29) $$|\mathcal{R}| < \tau^{-2} \log \frac{1}{\delta_1}.$$

We obtain a subset $I \subset \mathcal{R}$ satisfying

(10.30) $$\ell = |I| \sim \frac{\tau|\mathcal{R}|}{\log \frac{1}{\delta_1}}$$

and

(10.31) $$\sum_{n\in A''} \lambda''_n \prod_{r\in I} \big(1 + Re(b_r e(n\psi_r))\big) > (1 + \frac{\tau}{40}|I|)\lambda''(A'').$$

Recall (5.5)

$$\lambda'' = \lambda_{\theta, c(\frac{\kappa}{d})^2\varepsilon, c(\frac{\kappa}{d})^2 M}.$$

Define

$$\lambda^0 = \lambda_{\theta, c(\frac{\kappa}{d})^2(1+\frac{\delta_1^2}{d})\varepsilon, c(\frac{\kappa}{d})^2(1+\frac{\delta_1^2}{d})M} \tag{10.32}$$

and

$$\bar\lambda = \lambda_{\bar\theta, \bar\varepsilon, \bar M},$$

where

$$\begin{cases} \bar\theta = \big(\theta_1, \dots, \theta_d; \psi_r(r \in I)\big) \in \mathbb{T}^{d+\ell} \\ \bar\varepsilon = \Big(c\frac{\delta_1^6}{d^3}\varepsilon; \frac{c\delta_1^2}{\ell}(r \in I)\Big) \in \mathbb{R}_+^{d+\ell} \\ \bar M = c\frac{\delta_1^6}{d^3}M. \end{cases} \tag{10.33}$$

By construction, $\Lambda'' \subset \Lambda^0$ and

$$|\Lambda^0| < (1 + c\delta_1^2)|\Lambda''|. \tag{10.34}$$

If $n \in \Lambda''$ and $m \in \bar\Lambda$, then $n - m \in \Lambda^0$. Therefore,

$$\sum_m \bar\lambda_{n-m}\lambda_m^0 = \sum_m \lambda_{n-m}^0 \bar\lambda_m = \frac{1}{|\Lambda^0|} > (1 - c\delta_1^2)\frac{1}{|\Lambda''|} \quad \text{for } n \in \Lambda''$$

and

$$\sum_m \tilde\lambda_{n-m}\lambda_m^0 \geq (1 - c\delta_1^2)\lambda_n''. \tag{10.35}$$

From (10.31), (10.35),

$$\begin{aligned} \sum_{n \in A'', m} \bar\lambda_{n-m}\lambda_m^0 \prod_{r \in I}\big(1 + Re(b_r e(n\psi_r))\big) &> (1 - 2\delta_1^2)\Big(1 + \frac{\tau\ell}{40}\Big)\lambda''(A'') \\ &> \Big(1 + \frac{\tau\ell}{50}\Big)\lambda''(A''). \end{aligned} \tag{10.36}$$

Write the left side of (10.36) as

$$\sum_m \bar\lambda(A'' - m)\lambda_m^0 \prod_{r \in I}\big(1 + Re(b_r e(m\psi_r))\big) \tag{10.37}$$

$$+ \sum_{n \in A'', m} \bar\lambda_{n-m}\lambda_m^0 \Big[\prod_{r \in I}\big(1 + Re(b_r e(n\psi_r))\big) - \prod_{r \in I}\big(1 + Re(b_r e(m\psi_r))\big)\Big]. \tag{10.38}$$

We need to estimate (10.38). Expanding the product $\prod_{r \in I}$, we obtain

$$-\sum_{n \in A'', m} \bar\lambda_{n-m}\lambda_m^0 \sum_{\substack{S \subset I \\ \nu \in \{1,-1\}^S}} \Big(\prod_{r \in S}\frac{1}{2}b_r^{(\nu_r)}\Big)e\Big(m\sum_{r \in S}\nu_r\psi_r\Big)\Big(1 - e((n-m)\sum_{r \in S}\nu_r\psi_r)\Big),$$

where we let $b_r^{(1)} = b_r, b_r^{(-1)} \sim \bar{b}_r$.

It follows from (10.33) that if $\bar\lambda_k \neq 0$, then

$$\left\| k\Big(\sum \nu_r\psi_r\Big)\right\| < c\delta_1^2. \tag{10.39}$$

We now use the spectral synthesis property of the point $0 \in \mathbb{T}$.

Thus

$$1 - e^{2\pi i\theta} = \sum_{z\in\mathbb{Z}} \rho_z e^{2\pi i z\theta} \text{ for } \|\theta\| < c\delta_1^2, \tag{10.40}$$

where

$$\sum_z |\rho_z| \lesssim c\delta_1^2. \tag{10.41}$$

Substitution of (10.40) in (10.38) gives

(10.42)

$$-\sum_{z\in\mathbb{Z}} \rho_z \sum_{n\in A'',m} \bar\lambda_{n-m}\lambda_m^0 \sum_{\substack{S\subset I\\ \nu\in\{1,-1\}^S}} \Big(\prod \frac{1}{2} b_r^{(\nu_r)}\Big) e\Big(m + z(n-m)\Big(\sum \nu_r\psi_r\Big)\Big)$$
$$= -\sum_{z\in\mathbb{Z}} \rho_z \sum_{n\in A'',m} \bar\lambda_{n-m}\lambda_m^0 \prod_{r\in I} \big(1 + Re(b_r e\big(m + z(n-m)\psi_r)\big)\big)$$

and

$$(10.42) \leq \sum_{z\in\mathbb{Z}} |\rho_z| \sum_{n,m} \bar\lambda_{n-m}\lambda_m^o \prod_{r\in I} \big(1 + Re(b_r e(m + z(n-m)\psi_r))\big)\big)$$
$$\overset{(10.41)}{<} c\delta_1^2 \max_z \max_k \sum_m \lambda_m^0 \prod_{r\in I} \big(1 + Re\big(b_r e(zk\psi_r)e(m\psi_r)\big)\big). \tag{10.43}$$

From estimate (10.27), which remain valid with λ'' replaced by λ^0, and letting

$$\Omega_n = \prod_{r\in I} \big(1 + Re\big(b_r e(zk\psi_r)e(n\psi_r)\big)\big),$$

we conclude that

$$(10.38), (10.42) < c\delta_1^2. \tag{10.44}$$

Recalling (10.36), we find

$$(1 + c\delta_1^2). \max_m \bar\lambda(A'' - m) \geq \sum_m \bar\lambda(A'' - m)\lambda_m^0 \prod_{r\in I} \big(1 + Re\big(b_r e(m\psi_r)\big)\big)$$
$$> \Big(1 + \frac{\tau\ell}{50}\Big)\lambda''(A'') - c\delta_1^2. \tag{10.45}$$

Hence, we get a density increment for some translate $\bar{A} = A'' - m$ of A

$$\bar{\lambda}(\bar{A}) > \Big(1 + \frac{\tau\ell}{50}\Big)(\delta_1 - \kappa - c\delta_1^2) > \delta_1 - 2\kappa + \frac{\tau^2|\mathcal{R}|}{\log\frac{1}{\delta_1}}\delta_1. \tag{10.46}$$

We may now specify for which ranges of $\mathcal{R}$ we apply procedure (III) or (IV).

Requiring in (10.46)

$$\bar{\lambda}(\bar{A}) > \delta_1 + \kappa, \tag{10.47}$$

apply (IV) if

$$R \gtrsim \frac{\kappa \log\frac{1}{\delta_1}}{\tau^2\delta_1}. \tag{10.48}$$

In fact, assuming (10.48) and reducing $\mathcal{R}$ to a set of size (10.48), we obtain from (10.30)

$$\ell \sim \frac{\kappa}{\tau\delta_1} \tag{10.49}$$

(recall the assumption that $\tau\delta_1 < 10\kappa$).

The new Bohr set $\bar{\Lambda}$ is specified by (10.33).

If

$$R \lesssim \kappa\frac{\log\frac{1}{\delta_1}}{\tau^2\delta_1}, \tag{10.50}$$

we apply the procedure in §9.

§11. Summary

Let $\lambda = \lambda_{\theta,\varepsilon,M}$ s.t. for some translate A' of A,

$$\lambda(A') = \delta_1 \geq \delta. \tag{11.1}$$

We summarize the different scenarios of density increment occurring in the previous analysis. Let

$$\delta_1^2 \ll \kappa \ll \delta_1$$

be a fixed parameter.

If (5.18), there is an increment κ for a new Bohr set $\bar{\Lambda} = \bar{\Lambda}_{\bar{\theta},\bar{\varepsilon},\bar{M}}$ for which

$$\begin{cases} \bar{d} = d \\ \bar{\varepsilon} > c\Big(\frac{\kappa}{d}\Big)^2\varepsilon \\ \bar{M} > c\Big(\frac{\kappa}{d}\Big)^2 M \end{cases} \tag{11.2}$$

If (7.1), we obtained in §7 an increment $10^{-7}\delta_1$ for

$$\begin{cases} \bar d = d+1 \\ \bar\varepsilon > \Big(c\frac{\delta_1^9}{d^6}\Big(\sum \log\frac{1}{\varepsilon_j}\Big)^{-2}\varepsilon, \frac{1}{10}\Big) \\ \bar M > c\frac{\delta_1^9}{d^6}\Big(\sum\log\frac{1}{\varepsilon_j}\Big)^{-2} M \end{cases} \tag{11.3}$$

(cf. (7.9), (7.10)).

If $\tau\delta_1 > 10\kappa$, there is an increment κ for

$$\begin{cases} \bar d = d+1 \\ \bar\varepsilon = c\frac{\delta_1^2}{d}\Big(\big((c\kappa)/d\big)^2\varepsilon, 1\Big) \\ \bar M = c\frac{\delta_1^2}{d}\big((c\kappa)/d\big)^2 M \end{cases} \tag{11.4}$$

(cf. (8.8), (8.9)).

If $\tau\delta_1 \le 10\kappa$, there are 2 alternatives.

In the alternative considered in §10, (10.47) gives a density increment of κ for

$$\begin{cases} \bar d = d+\ell \\ \bar\varepsilon > \Big(c\frac{\delta_1^6}{d^3}\varepsilon; \underbrace{c\frac{\delta_1^2}{\ell}}_{\ell}\Big) \\ \bar M = c\frac{\delta_1^6}{d^3}M, \end{cases} \tag{11.5}$$

where by (10.49),

$$\ell < \kappa/\delta_1^2. \tag{11.6}$$

The alternative considered in §9 is of a different nature. Here we obtain a new density

$$\bar\delta > \frac{\delta_1}{2\tau\log\frac{1}{\delta_1}} > \frac{\delta_1^2}{20\kappa\log\frac{1}{\delta_1}} \tag{11.7}$$

for

$$\begin{cases} \bar d = d+R \\ \bar\varepsilon > \Big(c\frac{\delta_1^{20}}{d^8}\Big(\sum\log\frac{1}{\varepsilon_j}\Big)^{-2}\varepsilon, \underbrace{\frac{\delta_1^4}{3d}}_{\bar d - d}\Big) \\ \bar M > cd^{-8}\delta_1^{16}\Big(\sum\log\frac{1}{\varepsilon_j}\Big)^{-2}M, \end{cases} \tag{11.8}$$

recalling (9.19), (9.20), (10.8), (10.9), and where by (10.50),

$$R < \frac{\kappa\log\frac{1}{\delta_1}}{\tau^2\delta_1}. \tag{11.9}$$

Cases (11.2), (11.3), (11.4), (11.5) thus allow a density increment

$$\bar{\delta} = \delta_1 + \kappa \tag{11.10}$$

for

$$\begin{cases} \bar{d} < d + \frac{\kappa}{\delta_1^2} \\ \bar{\varepsilon} > c\frac{\delta_1^9}{d^6}\Big(\sum \log \frac{1}{\varepsilon_j}\Big)^{-2}\Big(\varepsilon, \underbrace{1, \dots 1,}_{\bar{d}-d}\Big) \\ \bar{M} > c\frac{\delta_1^9}{d^6}\Big(\sum \log \frac{1}{\varepsilon_j}\Big)^{-2} M. \end{cases} \tag{11.11}$$

Since we always assume $\delta > \frac{1}{\log N}, d < \log N, \varepsilon_j > \frac{1}{N}$, we rewrite (11.11) as

$$\begin{cases} \bar{d} < d + \frac{\kappa}{\delta_1^2} \\ \bar{\varepsilon} > \frac{1}{(\log N)^{20}}\Big(\varepsilon, \underbrace{1, \dots, 1}_{\bar{d}-d}\Big) \\ \bar{M} > \frac{1}{(\log N)^{20}} M. \end{cases} \tag{11.12}$$

We take

$$\kappa = \delta_1^{3/2}. \tag{11.13}$$

If, the alternative from §9 occurs, we obtain a new density

$$\bar{\delta} > \frac{\tau^{-1}\delta_1}{\log\log N} > \frac{\delta_1^{1/2}}{\log\log N} \tag{11.14}$$

for

$$\begin{cases} \bar{d} < d + \frac{\delta_1^{1/2}(\log\log N)}{\tau^2} \\ \bar{\varepsilon} > \frac{1}{(\log N)^{40}}\big(\varepsilon, \underbrace{1 \dots 1}_{\bar{d}-d}\big) \\ \bar{M} > \frac{M}{(\log N)^{40}} \end{cases} \tag{11.15}$$

for some

$$\delta_1 < \tau < 10\delta_1^{1/2}. \tag{11.16}$$

Next we explain the strategy.

Starting from $A \subset \{1, \dots, N\}$, $|A| = \delta N$ $(d = 0, M = N)$, we increase the density in Bohr sets according to procedures (I), (II), (IV). If, at some point, procedure (III) is required, we apply (III) and continue the process, taking each time $\kappa = \delta_1^2$ with at each step an increment of the number of generators by 1 and the density by at least δ_1^2.

Thus process (III) is applied at most once.

In the first stage, the parameters $\delta_\alpha, d_\alpha, \varepsilon_\alpha, M_\alpha$ evolve according to (11.10), (11.12)

$$\begin{cases} \delta_{\alpha+1} \geq \delta_\alpha + \delta_\alpha^{3/2} \\ d_{\alpha+1} < d_\alpha + \delta_\alpha^{-1/2} \\ \varepsilon_{\alpha+1} > \frac{1}{(\log N)^{20}}\big(\varepsilon_\alpha, \underbrace{1, \ldots 1}_{d_{\alpha+1}-d_\alpha}\big) \\ M_{\alpha+1} > \frac{M_\alpha}{(\log N)^{20}}. \end{cases} \tag{11.17}$$

Assume we encounter (III) at some step $\bar{\alpha}$. Then, according to (11.14), (11.15), for some

$$\delta_{\bar{\alpha}} < \tau \lesssim \delta_{\bar{\alpha}}^{1/2}, \tag{11.18}$$

we get

$$\delta_{\bar{\alpha}+1} > \frac{\delta_{\bar{\alpha}}}{\tau \log\log N} \tag{11.19}$$

$$\begin{cases} d_{\bar{\alpha}+1} < d_{\bar{\alpha}} + \frac{\delta_{\bar{\alpha}}^{1/2} (\log\log N)}{\tau 2} \\ \varepsilon_{\bar{\alpha}+1} > \frac{1}{(\log N)^{40}}\big(\varepsilon_{\bar{\alpha}}, \underbrace{1, \ldots 1}_{d_{\bar{\alpha}+1}-d_{\bar{\alpha}}}\big) \\ M_{\bar{\alpha}+1} > \frac{M_{\bar{\alpha}}}{(\log N)^{40}}. \end{cases} \tag{11.20}$$

For $\alpha > \bar{\alpha}$ (third stage),

$$\begin{cases} \delta_{\alpha+1} > \delta_\alpha + \delta_\alpha^2 \\ d_{\alpha+1} \leq d_\alpha + 1 \\ \varepsilon_{\alpha+1} > \frac{1}{(\log N)^{20}}(\varepsilon_\alpha, 1) \\ M_{\alpha+1} > \frac{M_\alpha}{(\log N)^{20}}. \end{cases} \tag{11.21}$$

The process terminates at $\alpha = \beta$ if

$$\varepsilon_\beta \sim 1 \tag{11.22}$$

or if (6.10) is violated, i.e.,

$$\log M_\beta \lesssim \sum_j \log \frac{1}{\varepsilon_{\beta,j}} + d_\beta \log\log N. \tag{11.23}$$

Assume (III) is not encountered. Then (11.17) is iterated (only the first stage appears). Starting from $\delta_0 = \delta$, the doubling appears in $\alpha \sim \delta^{-1/2}$ steps; and the number of generators is at most δ^{-1}. Certainly,

$$\varepsilon_{\alpha,j} > \frac{1}{(\log N)^{20\delta^{-1/2}}} \tag{11.24}$$

$$M_\alpha > \frac{N}{(\log N)^{20\delta^{-1/2}}}. \tag{11.25}$$

Since we reach (11.22) again in $\beta \sim \delta^{-1/2}$ steps, we need to ensure that

$$\log N \gg \frac{1}{\delta^{3/2}} \log\log N, \tag{11.26}$$

and this is satisfied for

$$\delta \gg \Big(\frac{\log\log N}{\log N}\Big)^{2/3}. \tag{11.27}$$

Next assume (III) occurs at some stage $\bar\alpha \lesssim \delta^{-1/2}$. Thus

$$\varepsilon_{\bar\alpha,j} > \frac{1}{(\log N)^{20\delta^{-1/2}}} \quad \text{for } 1 \le j \le d_{\bar\alpha} \lesssim \delta^{-1} \tag{11.28}$$

and

$$M_{\bar\alpha} > \frac{N}{(\log N)^{20\delta^{-1/2}}}. \tag{11.29}$$

According to (11.19), (11.20),

$$\begin{cases} \delta_{\bar\alpha+1} > \frac{\delta_{\bar\alpha}}{\tau \log\log N} \\ d_{\bar\alpha+1} \lesssim \delta^{-1} + \log\log N \frac{\delta_{\bar\alpha}^{1/2}}{\tau^2} \\ \varepsilon_{\bar\alpha+1} > \Big(\underbrace{(\log N)^{-20\delta^{-1/2}}}_{d_{\bar\alpha}}, \underbrace{(\log N)^{-40}}_{d_{\bar\alpha+1}-d_{\bar\alpha}}\Big) \\ M_{\bar\alpha+1} > \frac{N}{(\log N)^{20\delta^{-1/2}}}. \end{cases} \tag{11.30}$$

Continuing with the iteration (11.21) until reaching (11.22) clearly gives

$$\begin{cases} \beta \le \bar\alpha + 1 + C\delta_{\bar\alpha+1}^{-1} \lesssim \delta^{-1/2} + \frac{\tau\log\log N}{\delta_{\bar\alpha}} \\ d_\beta < d_{\bar\alpha+1} + C\delta_{\bar\alpha+1}^{-1} \lesssim \delta^{-1} + \delta_{\bar\alpha}^{1/2}\tau^{-2}\log\log N + \delta_{\bar\alpha}^{-1}\tau\log\log N \\ \varepsilon_\beta > \Big(\underbrace{(\log N)^{-20\delta^{-1/2}-(\log\log N)\tau\delta_{\bar\alpha}^{-1}}}_{d_{\bar\alpha}}, \underbrace{(\log N)^{-(\log\log N)\tau\delta_{\bar\alpha}^{-1}}}_{d_\beta - d_{\bar\alpha}}\Big) \\ M_\beta > \frac{N}{(\log N)^{10\delta^{-1/2}+(\log\log N)\tau\delta_{\bar\alpha}^{-1}}}. \end{cases} \tag{11.31}$$

Examining (11.23), using (11.31) and (11.18), we see that we need to ensure that (11.32)

$$\log N \gg (\log\log N)^2\delta^{-1/2} + \delta^{-1}(\log\log N)^2\delta^{-1/2} + (\log\log N)^3\Big(\frac{\delta_{\bar\alpha}^{1/2}}{\tau^2} + \frac{\tau}{\delta_{\bar\alpha}}\Big).\tau\delta_{\bar\alpha}^{-1}.$$

Hence

$$\log N \gg (\log\log N)^2\delta^{-3/2} + (\log\log N)^3\delta_{\bar\alpha}^{-3/2}$$

$$\log N \gg (\log\log N)^3.\delta^{-3/2},$$

and from which we obtain the final density condition

(11.33) $$\delta \gg \frac{(\log\log N)^2}{(\log N)^{2/3}}$$

Appendix

We prove Proposition $(*)$. Let us recall the statement

Proposition $(*)$. *Let (Ω, μ) be a probability space and $(\varphi_j)_{1\le j\le k}$ real functions on Ω satisfying*

(i) $$|\varphi_j| < 1/2 \qquad (1 \le j \le k)$$

and

(ii) $$\int \prod_{j=1}^{k}(1 + a_j\varphi_j)d\mu < 2 \quad \textit{for all } a_j \in [-1, 1].$$

Let $A \subset \Omega, \mu(A) = \delta$ and assume that

(iii) $$\int_A \varphi_j d\mu > \tau\delta \quad (1 \le j \le k).$$

Then

(iv) $$k < C\tau^{-2}\log\frac{1}{\delta}$$

and there is a subset $I \subset \{1, \dots, k\}$ such that

(v) $$|I| \sim \frac{\tau k}{\log\frac{1}{\delta}}$$

and

(vi) $$\int_A \prod_{j\in I}(1 + \varphi_j)d\mu > \delta(1 + \frac{\tau}{2}|I|).$$

Proof. Let $1 < p < 2, \bar\Omega = \{1,-1\}^k$, and write

$$(1)\quad \Big\| \int_A \prod_{j=1}^k (1+\sqrt{p-1}\,\omega_j\varphi_j)d\mu \Big\|_{L^2(\bar\Omega)} = \Big\{ \sum_{S\subset\{1,\dots,k\}} (p-1)^{|S|} \Big[\int_A \Big[\prod_{j\in S}\varphi_j\Big] d\mu\Big]^2 \Big\}^{1/2}.$$

By hyper-contractivity, the left side of (1) is bounded by

$$(2)\quad \Big\| \int_A \prod_{j=1}^k (1+\omega_j\varphi_j)d\mu \Big\|_{L^p(\Omega)} = \|T\mathcal{X}_A\|_{L^p(\bar\Omega)} \le \|T\|_{p\to p}\delta^{1/p},$$

where T is the operator

$$(3)\quad Tf(\omega) = \int f \prod_{j=1}^k (1+\omega_j\varphi_j)d\mu.$$

Obviously,

$$\|Tf\|_1 \le \int |f(x)| \Big[\int \prod_{j=1}^k (1+\omega_j\varphi_j)d\omega \Big] d\mu = \|f\|_{L^1};$$

hence

$$(4)\quad \|T\|_{1\to 1} \le 1.$$

By (ii),

$$\|Tf\|_\infty \le \Big[\int \prod_j (1+\omega_j\varphi_j)d\mu \Big] \|f\|_\infty \le 2\|f\|_\infty,$$

which implies that

$$(5)\quad \|T\|_{\infty\to\infty} \le 2.$$

Interpolation between (4), (5) gives

$$(6)\quad \|T\|_{p\to p} \le 2 \quad \text{for all } 1 \le p \le \infty.$$

Therefore from (1), (2), for $1 \le s \le k$,

$$(7)\quad \Big\{ \sum_{\substack{S\subset\{1,\dots,k\}\\ |S|=s}} \Big[\int_A \Big[\prod_{j\in S}\varphi_j\Big] d\mu \Big]^2 \Big\}^{1/2} \le 2\Big(\frac{1}{p-1}\Big)^{s/2} \delta^{1/p}.$$

Choosing

$$p = 1 + \frac{1}{\log 1/\delta},$$

we have that

$$\left\{ \sum_{\substack{S \subset \{1,\dots,k\} \\ |S|=s}} \Big[\int_A \prod_{j \in S} \varphi_j d\mu \Big]^2 \right\}^{1/2} < \Big(\log \frac{1}{\delta} \Big)^{s/2} \delta. \tag{8}$$

Taking $s = 1$ and recalling (iii), we have

$$\tau \delta \sqrt{k} < \Big(\log \frac{1}{\delta} \Big)^{1/2} \delta,$$

so

$$k < \Big(\log \frac{1}{\delta} \Big) \tau^{-2}, \tag{9}$$

which is (v).

The set $I \subset \{1, \dots, k\}, |I| = \ell = \eta k$ is chosen at random. Write

$$\int_A \prod_{j \in I} (1 + \varphi_j) d\mu = \mu(A) + \sum_{j \in I} \int_A \varphi_j d\mu + \sum_{2 \le s \le \ell} \sum_{\substack{S \subset I \\ |S|=s}} \int_A \prod_{j \in S} \varphi_j d\mu$$

$$\overset{\text{(iii)}}{\geq} \delta + \ell \tau \delta - \sum_{2 \le s \le \ell} \binom{\ell}{s}^{1/2} \Big[\sum_{\substack{S \subset I \\ |S|=s}} \Big(\int_A \prod_{j \in S} \varphi_j d\mu \Big) \Big|^2 \Big]^{1/2}. \tag{10}$$

We bound the last term in (10) by averaging over all sets $I \subset \{1, \dots, k\}$ of size $|I| = \ell$. Denote this averaging operator $\mathbb{E}_I$. Clearly,

$$\mathbb{E}_I \Big[\sum_{\substack{S \subset I \\ |S|=s}} \Big| \int_A \prod_{j \in S} \varphi_j d\mu \Big|^2 \Big]^{1/2} \le \Big\{ \sum_{\substack{S \subset \{1,\dots,k\} \\ |S|=s}} \frac{\binom{k-s}{\ell-s}}{\binom{k}{\ell}} \Big| \int_A \prod_{j \in S} \varphi_j d\mu \Big|^2 \Big\}^{1/2}$$

$$\overset{(8)}{<} \Big(\frac{\ell}{k - \ell} \Big)^{s/2} \Big(\log \frac{1}{\delta} \Big)^{s/2} \delta < \Big(2\eta \log \frac{1}{\delta} \Big)^{s/2} \delta. \tag{11}$$

Hence

$$\mathbb{E}_I \Big\{ \int_A \prod_{j \in I} (1 + \varphi_j) d\mu \Big\} \ge \delta \Big(1 + \eta k \tau - \sum_{2 \le s \le \ell} \Big(\frac{\eta^2 k \log \frac{1}{\delta}}{s} \Big)^{s/2} \Big). \tag{12}$$

Take

$$\eta = c \frac{\tau}{\log \frac{1}{\delta}}. \tag{13}$$

From (9),

$$\eta^2 k \log \frac{1}{\delta} = c\frac{\tau^2 k}{\log \frac{1}{\delta}} < 1;$$

and therefore,

$$(12) > \delta\Big(1 + \eta k\tau - \eta^2 k \log \frac{1}{\delta}\Big) > \delta\Big(1 + \frac{1}{2}\eta k \tau\Big), \tag{14}$$

which is (vi). This proves Proposition $(*)$. □

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