The Book Review Column[1]

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Reviews [2] of THREE books on Fair Division of Resources

Fair Division (From Cake Cutting to Dispute Resolution)
by Steven Brams and Alan Taylor
Published by Cambridge University Press, 272 pages, 1996
Softcover, $21.95
ISBN number 0-521 R5539-0

AND

Cake-Cutting Algorithms(Be Fair if You Can)
by Jack Robertson and William Webb
Published by A.K. Peters, 177 pages, 1998
Hardcover, $38.00
ISBN number 1-5688-10766

AND

The Win-Win Solution
by Steven Brams and Alan Taylor
Published by Norton, 177 pages, 1999
Softcover, $13.95
ISBN number 0-393-04729-6

AND

Fair Allocation
Proceedings of Symposia in Applied Mathematics, Vol 33
Edited by H. Peyton Young
Published by AMS, 1986,170 pages
Softcover, $31.00
ISBN number 0-8218-0094-9

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1 Overview

The following problem arises in many real world situations: How do you divide a quantity among n people in a fair manner? To make this question rigorous we have to know more about the item, the people, and the notion of fairness. The following questions need to be raised.

1. Is the quantity continuous (like a cake) or discrete (like a set of items in an inheritance)?
2. What is each players valuation? (For example Bill values the chocolate part of the cake.)
3. What do the players know about the other players valuations?
4. If a player thinks (in his measure of value) that he got $\geq \frac{1}{n}$ of the total value, but that someone else got more than he got, is that fair? (Say Bob, Carol, Ted, and Alice split a cake such that Bob thinks he got $\frac{1}{3}$, but he also thinks that Carol got $\frac{1}{12}$, Ted got $\frac{1}{12}$, and Alice got $\frac{1}{2}$. All the other players think that they each got $\frac{1}{4}$. Is this fair?)
5. Is there a trusted referee?
6. If the item is continuous then can we use continuous methods? For example a knife could be passed over the cake until someone yells STOP, and whoever yells STOP gets the piece.
7. If the item is actually a set of discrete items then what methods are allowed? Auctions? Voting?
8. If the method being used is discrete then how many operations are allowed? For example, we might want to minimize the number of cuts when dividing a cake.
9. Is the item something people want like cake or something people do not want like chores?
10. Is an approximate solution good enough? How approximate?
11. What if the players want unequal shares? For example both Alice and Bob can agree that they want Alice to get $\frac{2}{3}$ and Bob to get $\frac{1}{3}$.
12. Given a way to divide the item(s) is there another division where at least one player is better off and nobody else is worse off?
13. Assuming that all parties know each others valuations, do all parties think that they got the higher (or tied) fraction of goods then the other parties. For example, if after a divorce Alice thinks she god 60% of the estate, Bob thinks he got 90% of the estate, and Alice knows how Bob's feels, this might not be considered fair.
14. Is the protocol easy to use?

The three books under review look at mathematically rigorous versions of some of these problems mentioned above. To give a flavor for the subject we present two definitions and two protocols.

A division of a good among n people is *proportional* if each person thinks they got at least $\frac{1}{n}$. A division of a good among n people is *envy-free* if each person believes his piece is at least as big as everyone elses.

Here is a discrete envy-free protocol among Bob, Carol, and Alice to divide a cake in three pieces. It is due independently to Selfridge (1960) and Conway (1993)[3] As is the convention we write the protocol and then put in parenthesis what the player should do in his or her self interest. A phrase like "Bob takes a piece" actually means "Bob takes the piece that is largest in his valuation."

1. Bob cuts the cake into three pieces. (Equal pieces.)
2. Carol trims at most one piece. The trimming is set aside and called T. (Carol trims the unique largest piece, if it exists, to create a tie for largest.)
3. Alice takes a piece.
4. If Alice did not take the trimmed piece then Carol must take it. If Alice did take the trimmed piece then Carol can take whichever of the two pieces left that she wants.
5. Bob gets the piece that is left. Note that the cake minus T has been split in an envy-free way.
6. Assume that Carol got the trimmed piece (if Alice did then the procedure is similar). Note that if Carol gets all or part of the trimming T then Bob does not mind since he thinks Carol's piece plus T is equal to his piece. Alice cuts T into three (equal) pieces.
7. Carol takes a piece.
8. Bob takes a piece. Note that Bob does not mind if Carol got a bigger piece of T then he did.
9. Alice gets the remaining piece.

It is not hard to show that the above protocol is envy-free. Note that it takes at most five cuts. There is a discrete envy-free protocol for $n = 4$ and even for general n; however, the number of cuts they take depends on the players valuations of the cake and is finite but unbounded. For example, there is a valuation that the four players could have so that 10^{100} cuts are needed. It is an open problem to find a bounded protocol for envy-free division with $n = 4$.

We now present a continuous proportional protocol that works for general n but is not envy-free.

1. The knife is passed over the cake left to right until some player yells STOP. (That player thinks the piece between the left end and the knife is $1/n$.)
2. The player who yells stop gets the piece between the knife and the left end. (Ties are decided arbitrarily.) He should be happy since he thinks its $1/n$ of the cake. The rest of the players should be happy because since they didn't yell STOP they thing that at least $1 - 1/n$ is left.
3. Repeat the protocol with the $n - 1$ players left and with the cake that is left.

[3] The books under review reference Selfridge and Conway as I did above; hence I assume there results were never formally published.

2 Fair Division (From Cake Cutting to Dispute Resolution)

This book deals with both the continuous and the discrete case. Both proportional and envy-free solutions are considered. Both discrete and moving-knife solutions are considered. Real world examples are discussed (e.g., negotiating the Panama Canal Treaty). In addition there are full chapters on voting, auctions, and the divide-the-dollar game (how to get two parties to divide a dollar fairly).

The book is well written and would be appropriate for an interdisciplinary undergraduate course. Students from mathematics, computer science, economics, and sociology would be the audience. No particular math background is needed; however, mathematics maturity is required. There are no exercises, which might limit its usefulness as a textbook for a standard HW-exam course; however, there are many references to the Economics and Political Science literature, which would allow for a course where students write papers.

The book is well organized. Most of the chapters are on a well defined topic (e.g., "Envy-free division for $n \geq 3$"). There are many footnotes which allow for easy personal reading since you can choose which footnotes to skip.

3 Cake-Cutting Algorithms (Be Fair if You Can)

This book deals mostly with the continuous case, i.e., cake cutting. They assume that the players do not know each others valuations. Both proportional and envy-free solutions are considered. Both discrete and moving-knife solutions are considered. Upper and lower bounds on the number of cuts are also discussed. Some mathematics of interest is used: Ramsey partitions and bipartite graph matching.

The book is well written and would be appropriate for an interdisciplinary undergraduate course. Each chapter includes exercises. Hence this could be a real course with HW and exams. Students from mathematics, computer science, economics, and sociology would be the audience. No particular math background is needed; however, mathematics maturity is required.

In the preface the author states "The first six chapters provide a leisurely survey of the problems, Chapter 7 is a quick reference which summarizes known results, while the last four chapters contain technical proofs for previously introduced problems." This format is good for students and for planning a course. The first seven chapters could form a course. A more mature class could do the first seven chapters and some subset of $\{8, 9, 10, 11\}$. For your own personal reading it can be inconvenient to flip back and fourth many times..

4 Differences between *Fair Division* and *Cake Cutting*

The following topics are in *Fair Division* but not in *Cake Cutting*.

1. Dividing a set of discrete objects (Chapters 3,5).
2. Dividing a dollar (Chapter 8).
3. Auctions (Chapter 9).
4. Voting (Chapter 10).

5. What if a player knows the other one's valuation. (Section 1.4. There is not that much rigorous to say.)

6. Applications. (Parts of Chapters 1 and 4, but also scattered throughout.)

The following topics are in *Cake Cutting* but not in *Fair Division*

1. The number of cuts needed to divide a cake proportionally (Chapters 2,8,9).

2. Unequal shares. This uses Ramsey Partitions. (Chapters 3, 11).

3. Exact-fair division where everyone thinks they have *exactly* $1/n$. (Parts of Chapters 5 and 10.)

4. Dividing up chores (Section 5.5).

5. An application of Hall's theorem (Chapter 6 which is short).

6. Exercises and suggested projects.

5 The Win-Win Solution

This book is for the layperson; however, readers such as ourselves can still benefit from it. This book looks at a few schemes for dividing a set of discrete goods among two people and then applies them to very real world scenario. The Camp David accords, the (first) Trump divorce, the Spately Island disupte, the debate over the Clinton-Dole debates, and the Princess Di divore are all discussed in detail. Other disputes are discussed in passing. By applying a technique called *Adjusted Winner* one obtains a fair settlement which seems to match the real settelments reached. One catch is that the participants have to know how much they value various goods and report on this honestly. The estimates given in the book seem reasonable.

The main technique discussed is Adjusted Winner. Each particpant intially (and privately) distributes 100 points over all the goods available. Then in phase one the goods are distributed (roughly) based on who wants what more. A second phase transfers goods from one participant to another so that each participant gets at least 50 points in his estimation.

This book has much discussion of how to try to take advantage of knowledge of your opponents valuation. They conclude that using this knowledge to mispresent yourself for an advantage could is very dangerous— you might get far less than you would being honest.

6 Fair Allocation

This book is a collection of six self-contained articles on fairness issues that arise in politics and public policy. Each article begins with a real world problem, models it mathematically, and discusses solutions. Real data is presented. Each article is well written though a bit dense. A good senior math major or theory-inclined computer science major could read it. The authors do not have any political viewpoint to push which, while not surprising, is refreshing during this election year. This book was published in 1986. For a survey of some newer results on this topic see [1].

The articles tend to have more definitions and examples then theorems. This is probably caused by the clash of clean mathematics with the dirty world of politics. The book is from 1985 hence there has been some new material on some of these issues since then. We briefly describe each chapter by describing the problem they deal with. We could end each review with the line "This article survey's known solutions and the problems they may have."

6.1 The Apportionment Problem

The U.S. Constitution dictates that the number of representatives a state sends to the house or representatives is proportional to its population. This can lead to a fractional number of delegates ("The 3.41 of the New York delegates voted NO.") While it would seem that rounding off would solve the problem, this may lead to problems.

6.2 Inequality Measures

Politicians often say things like "That tax plan will further widen the gap between rich and poor." Such statements are not rigorous. How do you measure the gap?

6.3 Cost Allocation

If several towns are going to cooperate and build a common water treatment plant then what is the best way to divide the cost? There are many factors such as population and the cost savings achieved by pooling the efforts.

6.4 The Allocation of Debts and Taxes

When dividing an estate among several people what is the fairest allocation? The people may have different rights to the estate. For example perhaps the estate should be split $(1/2, 1/6, 1/12, 1/12)$. If the estate is small you may not want to invoke those rights since it could leave someone with (essentially) nothing. But this leads to another question— what is "small"?

6.5 Voting

If there are two candidates competing for an office then having a vote and having the majority candidate win is fine. If you have three candidates then several problems arise: (1) It is possible that no candidate gets the majority. (2) People may not vote honestly. For example one often hears "I want to vote for Harry Browne (Libertarian Party) but I know he can't win so instead I'll vote for George W. Bush." (3) If people rank their choices you may have that a majority prefer A to B, a majority prefer B to C and a majority prefer C to A. So what is the best scheme? A well known theorem by Arrow shows that you cannot have an voting system that satisfies a (reasonable) set of criteria. However, there are systems with various pros and cons that are worth looking at.

6.6 Auctions and Competitive Bidding

There are many types of auctions and many types of criteria for fairness. For example, lets say we want to maximize profit for the auctioneer. We might want to use a second-price auction where the winner pays the second highest bid. This might increase profit since the bidders can be more aggressive.

7 Opinion

I recommend both *Fair Division* and *Cake Cutting* for a course on this topic. You can reread the section of this review on the differences to decide which one is right for you. Whichever one you use as a text I recommend buying the other one for yourself. I also recommend *Fair Allocation* for personal reading, though not as a text. I recommend *The Win-Win Solution* as background for a course or to give to a friend or relative who is interested in these issues but not in the mathematics.

Above all else I recommend that theoretical computer scientists take an interest in this area. The problems studied are interesting and use the kind of mathematics we are familiar with (discrete math).

References

[1] E. Hemaspaandra and L. Hemaspaandra. Computational politics: Electoral systems. In *Proceedings of the 25th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 64–83. Springer-Verlag *Lecture Notes in Computer Science #1893*, August/September 2000.