The Canonical VDW Theorem
An Exposition by Bill Gasarch

We first recall the following version of van der Waerden's theorem.
VDW For every $k \ge 1$ and $c \ge 1$ for every c-coloring $COL : [\mathbb{N}] \rightarrow [c]$ there exists a monochromatic k-AP. In other words there exists a, d such that

$$COL(a) = COL(a + d) = \cdots = COL(a + (k-1)d).$$

What if we use an infinite number of colors instead of a finite number of colors? Then the theorem is false as the coloring $COL(x) = x$ shows. However, in this case, we may get something else.

Def 0.1 Let $k \in \mathbb{N}$. Let COL be a coloring of $\mathbb{N}$ (which may use a finite or infinite number of colors). A *rainbox k-AP* is an arithmetic sequence $a, a+d, a+2d, \ldots, a+(k-1)d$ such that all of these are colored *differently.*

The following is the *Canonical van der Waerden's theorem.* Erdos and Graham [**?**] claim that it follows from Szemerëdi's theorem on density. Later Prömel and Rödl [**?**] obtained a proof that used the Gallai-Witt theorem.

Theorem 0.2 *Let $k \in \mathbb{N}$. Let $COL : \mathbb{N} \rightarrow \mathbb{N}$ be a coloring of the naturals. One of the following two must occur.*

- *There exists a monochromatic k-AP.*
- *There exists a rainbox k-AP.*

Proof:

Let COL^* be the following *finite* coloring of $\mathbb{N} \times \mathbb{N}$. Given (a, d) look at the following sequence

$$(COL(a), COL(a + d), COL(a + 2d), \ldots, COL(a + (k-1)d)).$$

This coloring partitions the numbers $\{0, \ldots, k-1\}$ in terms of which coordinates are colored the same. For example, if $k = 4$ and the coloring was (R, B, R, G) then the partition is $\{\{0, 2\}, \{1\}, \{3\}\}$. We map (a, d) to the partition induced on $\{0, \ldots, k-1\}$ by the coloring. There are only a finite number of such partitions. (The Stirling numbers of the second kind are $S(k, L)$ are the number of ways to partition k numbers into L nonempty sets. The Bell numbers are $B_k = \sum_{L=1}^{k} S(k, L)$. The actual number is colors is B_k.)

Example 0.3

1. Let $k = 10$ and assume

   $$(COL(a), COL(a+d), \ldots, COL(a+(9d)) = (R, Y, B, I, V, Y, R, B, B, R).$$

   Then (a, d) maps to $\{\{0, 6, 9\}, \{1, 5\}, \{2, 7, 8\}, \{3\}, \{4\}, \}$.

2. Let $k = 6$ and assume

   $$(COL(a), COL(a + d) \ldots, COL(a + (5d)) = (R, Y, B, I, V, Y).$$

   Then (a, d) maps to $\{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$.

Let M be a constant to be picked later. By the Gallai-Witt theorem there exists a, d, D such that all of the following are the same COL^*

$$\{(a + iD, d + jD) \mid -M \le i, j \le M\}.$$

There are two cases.

Case 1: $COL^*(a, d)$ is the partition of every element into its own class. This means that there is a rainbow k-AP and we are done.

Case 2: There exists x, y such that $COL^*(a, d)$ is the partition that puts $a+xd$ and $a+yd$ in the same class. More simply, $COL(a+xd) = COL(a+yd)$. Since for all $-M \le i, j \le M$,

$$COL^*(a, d) = COL^*(a + iD, d + jD).$$

we have that, for all $-M \le i, j \le M$,

$$COL(a + iD + x(d + jD)) = COL(a + iD + y(d + jD)).$$

Assume that $COL(a + xd) = COL(a + yd) = \text{RED}$. Note that we do not know $COL(a + iD + x(d + jD))$ or $COL(a + iD + y(d + jD))$, but we do know that they are the same.

We want to find the (i, j) with $-M \le i, j \le M$ such that $COL^*(a + iD, d + jD)$ affects $COL(a + xd)$.

Note that

if

$$a + xd = a + iD + x(d + jD)$$

then

$$xd = iD + xd + xjD$$

$$0 = iD + xjD$$

$$0 = i + xj$$

$$i = -xj.$$

Hence we have that

$$a + xd = (a - xj)D + x(d + jD).$$

So what does this tell us? For all $-M \le i, j \le M$,

$$COL(a + iD + x(d + jD)) = COL(a + iD + y(d + jD)).$$

Let $i = -xj$ and you get

$$COL(a - xjD + x(d + jD)) = COL(a - xjD + y(d + jD)).$$

$$\text{RED} = COL(a + xd) = COL(a + yd + j(yD - xD)).$$

This holds for $-M \le j \le M$. Looking at $j = 0, 1, \ldots, k-1$, and letting $A = a + yd$ and $D' = yD - xD$, we get

$$COL(A) = COL(A{+}D') = COL(A{+}2D') = \cdots = COL(A{+}(k{-}1)D') = \text{RED}.$$

This yields an monochromatic k-AP.

What value do we need for M? We want $j = 0, 1, \ldots, k-1$. We want $i = -xj$. We know that $x \le k-1$. Hence it suffices to take $M = (k-1)^2$. ∎

Note 0.4 We used the two-dimensional VDW to prove the one-dimensional canonical VDW. For all d there is a d-dimensional canonical VDW, and it is proven using the $d+1$-dimensional VDW. The actual statement is somewhat complicated. The interested reader can see [**?**].

References

[1] P. Erödos and R. Graham. *Old and New Problems and results in Combinatorial Number Theory.* Academic Press, 1980. book 28 in a series called L'Enseignement Math.. This book seems to be out of print.

[2] H. J. Rödl and V. Prömel. An elementary proof of the canonizing version of Gallai-Witt's theorem. *JCTA*, 42:144–149, 1986.