The Canonical VDW Theorem
An Exposition by Bill Gasarch

We first recall the following version of van der Waerden’s theorem.

VDW For every \( k \geq 1 \) and \( c \geq 1 \) for every \( c \)-coloring \( \text{COL} : \mathbb{N} \rightarrow [c] \) there exists a monochromatic \( k \)-AP. In other words there exists \( a, d \) such that

\[
\text{COL}(a) = \text{COL}(a + d) = \cdots = \text{COL}(a + (k - 1)d).
\]

What if we use an infinite number of colors instead of a finite number of colors? Then the theorem is false as the coloring \( \text{COL}(x) = x \) shows. However, in this case, we may get something else.

**Def 0.1** Let \( k \in \mathbb{N} \). Let \( \text{COL} \) be a coloring of \( \mathbb{N} \) (which may use a finite or infinite number of colors). A *rainbow \( k \)-AP* is an arithmetic sequence \( a, a + d, a + 2d, \ldots, a + (k - 1)d \) such that all of these are colored differently.

The following is the *Canonical van der Waerden’s theorem*. Erdos and Graham [?] claim that it follows from Szemerédi’s theorem on density. Later Prőmel and Rödl [?] obtained a proof that used the Gallai-Witt theorem.

**Theorem 0.2** Let \( k \in \mathbb{N} \). Let \( \text{COL} : \mathbb{N} \rightarrow \mathbb{N} \) be a coloring of the naturals. One of the following two must occur.

- There exists a monochromatic \( k \)-AP.
- There exists a rainbow \( k \)-AP.

**Proof:**

Let \( \text{COL}^* \) be the following *finite* coloring of \( \mathbb{N} \times \mathbb{N} \). Given \( (a, d) \) look at the following sequence

\[
(\text{COL}(a), \text{COL}(a + d), \text{COL}(a + 2d), \ldots, \text{COL}(a + (k - 1)d)).
\]

This coloring partitions the numbers \( \{0, \ldots, k - 1\} \) in terms of which coordinates are colored the same. For example, if \( k = 4 \) and the coloring was \((R, B, R, G)\) then the partition is \( \{\{0, 2\}, \{1\}, \{3\}\} \). We map \( (a, d) \) to the partition induced on \( \{0, \ldots, k - 1\} \) by the coloring. There are only a finite number of such partitions. (The Stirling numbers of the second kind are \( S(k, L) \) are the number of ways to partition \( k \) numbers into \( L \) nonempty sets. The Bell numbers are \( B_k = \sum_{L=1}^{k} S(k, L) \). The actual number is colors is \( B_k \).)
Example 0.3

1. Let \( k = 10 \) and assume

\[
(COL(a), COL(a+d), \ldots, COL(a+(9d))) = (R, Y, B, I, V, Y, R, B, B, R).
\]

Then \((a, d)\) maps to \({\{0, 6, 9\}, \{1, 5\}, \{2, 7, 8\}, \{3\}, \{4\}}\).

2. Let \( k = 6 \) and assume

\[
(COL(a), COL(a + d) \ldots , COL(a + (5d))) = (R, Y, B, I, V, Y).
\]

Then \((a, d)\) maps to \({\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}}\).

Let \( M \) be a constant to be picked later. By the Gallai-Witt theorem there exists \( a, d, D \) such that all of the following are the same \( COL^* \)

\[
\{(a + iD, d + jD) \mid -M \leq i, j \leq M\}.
\]

There are two cases.  

**Case 1:** \( COL^*(a, d) \) is the partition of every element into its own class. This means that there is a rainbow \( k \)-AP and we are done.  

**Case 2:** There exists \( x, y \) such that \( COL^*(a, d) \) is the partition that puts \( a + xd \) and \( a + yd \) in the same class. More simply, \( COL(a+xd) = COL(a+yd). \)

Since for all \(-M \leq i, j \leq M, \)

\[
COL^*(a, d) = COL^*(a + iD, d + jD).
\]

we have that, for all \(-M \leq i, j \leq M, \)

\[
COL(a + iD + x(d + jD)) = COL(a + iD + y(d + jD)).
\]

Assume that \( COL(a + xd) = COL(a + yd) = RED. \) Note that we do not know \( COL(a + iD + x(d + jD)) \) or \( COL(a + iD + y(d + jD)) \), but we do know that they are the same.

We want to find the \((i, j)\) with \(-M \leq i, j \leq M\) such that \( COL^*(a + iD, d + jD) \) affects \( COL(a + xd). \)

Note that  
if

\[
a + xd = a + iD + x(d + jD)
\]
then

\[ xd = iD + xd + xjD \]

\[ 0 = iD + xjD \]

\[ 0 = i + xj \]

\[ i = -xj. \]

Hence we have that

\[ a + xd = (a - xj)D + x(d + jD). \]

So what does this tell us? For all \(-M \leq i, j \leq M\),

\[ COL(a + iD + x(d + jD)) = COL(a + iD + y(d + jD)). \]

Let \(i = -xj\) and you get

\[ COL(a - xjD + x(d + jD)) = COL(a - xjD + y(d + jD)). \]

\[ \text{RED} = COL(a + xd) = COL(a + yd + j(yD - xD)). \]

This holds for \(-M \leq j \leq M\). Looking at \(j = 0, 1, \ldots, k - 1\), and letting \(A = a + yd\) and \(D' = yD - xD\), we get

\[ COL(A) = COL(A + D') = COL(A + 2D') = \cdots = COL(A + (k-1)D') = \text{RED}. \]

This yields an monochromatic \(k\)-AP.

What value do we need for \(M\)? We want \(j = 0, 1, \ldots, k - 1\). We want \(i = -xj\). We know that \(x \leq k - 1\). Hence it suffices to take \(M = (k - 1)^2\).

\[ \square \]

\textbf{Note 0.4} We used the two-dimensional VDW to prove the one-dimensional canonical VDW. For all \(d\) there is a \(d\)-dimensional canonical VDW, and it is proven using the \(d+1\)-dimensional VDW. The actual statement is somewhat complicated. The interested reader can see [?].
References
