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Note

An Elementary Proof of the Canonizing Version of Gallai–Witt's Theorem

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1. INTRODUCTION

A homothetic mapping (homothety) of the t-dimensional lattice grid $\mathbb{N}^t$ is a mapping h: $\mathbb{N}^t \to \mathbb{N}^t$ of the form $h(\mathbf{b}) = \mathbf{a} + d\mathbf{b}$, where $\mathbf{a} \in \mathbb{N}^t$ is a translation vector and d is a positive integer describing a dilatation.

A multidimensional version of van der Waerden's theorem on arithmetic progressions is independently due to Gallai and to Witt (for general references see [5]). It asserts that for every mapping Δ: $\{0,..., n-1\}^t \to \{0, 1\}$, where $n \geqslant n(t, m)$ is sufficiently large, there exists a homothety h: $\mathbb{N}^t \to \mathbb{N}^t$ such that $\Delta(h(\mathbf{b})) = \Delta(h(\mathbf{c}))$ for all $\mathbf{b}, \mathbf{c} \in \{0,..., m-1\}^t$.

A canonizing version of this theorem was proved by Deuber, Graham, Prömel, and Voigt [1]. Let $U \subseteq \mathbb{Q}^t$ be a linear subspace of the t-dimensional vector space over the rationals. Let Δ_U: $\mathbb{N}^t \to \mathbb{N}$ be a mapping with the property that $\Delta_U(\mathbf{b}) = \Delta_U(\mathbf{c})$ iff $\mathbf{b} - \mathbf{c} \in U$. Of course, Δ_U acts constantly on each coset of U and different cosets get different images.

Obviously, $\Delta_U(h(\mathbf{b})) = \Delta_U(h(\mathbf{c}))$ iff $\Delta_U(\mathbf{b}) = \Delta_U(\mathbf{c})$ for every homothety. Thus, Δ_U induces the same pattern on all homothetic copies of $\{0,..., m-1\}^t$.

A vector $\mathbf{b} \in \mathbb{Q}^t$ is called *admissible* for $S \subseteq \mathbb{N}^t$ iff there exists $\mathbf{a} \in \mathbb{Q}^t$ such that the affine line $\{\mathbf{a} + \lambda\mathbf{b} \mid \lambda \in \mathbb{Q}\}$ intersects S in at least two points. Let

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$\mathscr{A}(S)$ denote the set of linear subspaces of $\mathbb{Q}^t$ possessing a basis of admissible vectors. Additionally the null-space $\{\mathbf{0}\}$ belongs to $\mathscr{A}(S)$.

Note that $\Delta_{\{\mathbf{0}\}}$ is an one-to-one mapping. Furthermore for every two different subspaces U and V in $\mathscr{A}(S)$ the partitions on S which are induced by Δ_U and Δ_V are different. Hence the following canonizing version of the Gallai–Witt theorem is best possible:

THEOREM [1]. *Let $S \subseteq \mathbb{N}^t$ be a finite set. Then there exists a finite set $T \subseteq \mathbb{N}^t$ such that for every mapping Δ: $T \to \mathbb{N}$ there exists a homothety h: $\mathbb{N}^t \to \mathbb{N}^t$ and a linear subspace $U \in \mathscr{A}(S)$ with the property that $\Delta(h(\mathbf{b})) = \Delta(h(\mathbf{c}))$ iff $\mathbf{b} - \mathbf{c} \in U$ for every $\mathbf{b}, \mathbf{c} \in S$.*

The original proof is based on Fürstenberg and Katznelson's [3] density version of the Gallai–Witt result. Since Fürstenberg and Katznelson use heavy ergodic tools, the question remained open (cf. [1, 2, 4]) to find an elementary proof of the canonizing version of Gallai–Witt's theorem.

The aim of the note is to give such an elementary proof. As it turns out, a slight modification of this proof also yields a canonization theorem due to Spencer [6] which characterizes the canonical partitions of finite subsets of $\mathbb{R}^t$ with respect to the group of homotheties acting on $\mathbb{R}^t$.

2. Proof of Theorem

Put $n^t = \{0, ..., n-1\}^t$. The main tool for proving the theorem is the following:

LEMMA. *Let t, m be positive integers. Then there exists a positive integer $n = n(t, m)$ such that for every mapping Δ: $n^t \to \mathbb{N}$ there exists a homothety h: $\mathbb{N}^t \to \mathbb{N}^t$ such that for every line $L \in \mathscr{A}(m^t)$ the following is valid:*

if $\quad \Delta(h(\mathbf{y}_0)) = \Delta(h(\mathbf{y}_1))$ for some $\mathbf{y}_0, \mathbf{y}_1 \in m^t$ satisfying $\mathbf{y}_1 - \mathbf{y}_0 \in L$,

then $\quad \Delta(h(\mathbf{z}_0)) = \Delta(h(\mathbf{z}_1))$ for every $\mathbf{z}_0, \mathbf{z}_1 \in m^t$ satisfying $\mathbf{z}_1 - \mathbf{z}_0 \in L$.

First, we show how the theorem can be deduced from the lemma: Without loss of generality let $S = k^t$ for some nonnegative integer $k = \{0, ..., k-1\}$. Assume that the assertion of the lemma holds for some $m = m(k)$ which is sufficiently large with respect to k. Let $\{\mathbf{x}_0, ..., \mathbf{x}_{s-1}\} \subseteq k^t$ be a maximal linear independent set (considered as a subset of $\mathbb{Q}^t$) with the property that

$\Delta(\mathbf{x}_i)=\Delta(\mathbf{0})$ for every $i\in s$ and let X be the linear subspace of $\mathbb{Q}^t$ generated by $\{\mathbf{x}_0,..., \mathbf{x}_{s-1}\}$. We claim that

$$\Delta\restriction(X\cap k^t)=\text{const.} \tag{1}$$

Assuming (1), from the lemma it follows that $\Delta\restriction((\mathbf{b}+X)\cap k^t)$ is constant for every coset $\mathbf{b}+X$. Thus, since $\{\mathbf{x}_0,..., \mathbf{x}_{s-1}\}$ is maximal independent we can infer the theorem.

To prove (1) let $\mathbf{z}\in X\cap k^t$. Then there exist $\lambda_0,..., \lambda_{s-1}\in\mathbb{Q}$ such that $\mathbf{z}=\sum_{i=0}^{s-1}\lambda_i\mathbf{x}_i$. Furthermore there exists (a minimal) $p\in\mathbb{N}$ such that $p\lambda_i\in\mathbb{Z}$ for every $i\in s$. For $m=m(k)$ large enough, we have $\sum_{i=0}^{s-1}p\cdot|\lambda_i|\cdot\mathbf{x}_i\in m^t$. Hence, also $p\mathbf{z}\in m^t$. Note that $\Delta(p\mathbf{z})=\Delta(\mathbf{0})$ implies $\Delta(\mathbf{z})=\Delta(\mathbf{0})$. Thus, it remains to show that

$$\Delta(p\mathbf{z})=\Delta(\mathbf{0}). \tag{2}$$

We do this by induction on the length of the basis representation of $\mathbf{z}$. If $p\mathbf{z}=p\lambda_0\mathbf{x}_0$ then (2) follows from $\Delta(p\lambda_0\mathbf{x}_0)=\Delta(\mathbf{x}_0)=\Delta(\mathbf{0})$. Thus, assume that for all $p\mathbf{z}=\sum_{i=0}^{r-1}p\lambda_i\mathbf{x}_i\in m^t$ for some $r<s$, it holds that

$$\Delta\left(\sum_{i=0}^{r-1}p\lambda_i\mathbf{x}_i\right)=\Delta(\mathbf{0}). \tag{3}$$

Let $p\mathbf{z}=\sum_{i=0}^{r}p\lambda_i\mathbf{x}_i$. Note that from (3), the lemma and the fact that $\Delta(\mathbf{x}_r)=\Delta(\mathbf{0})$ it follows that

$$\Delta\left(\sum_{i=0}^{r-1}p\cdot|\lambda_i|\cdot\mathbf{x}_i+p\cdot|\lambda_r|\cdot\mathbf{x}_r\right)=\Delta\left(\sum_{i=0}^{r-1}p\cdot|\lambda_i|\cdot\mathbf{x}_i\right)=\Delta(\mathbf{0}).$$

Assume that for some l, where $0<l\leqslant r$, it is valid that $\Delta(\sum_{i=0}^{l}p|\lambda_i|\,\mathbf{x}_i+\sum_{i=l+1}^{r}p\lambda_i\mathbf{x}_i)=\Delta(\mathbf{0})$. Then, using $\Delta(\mathbf{x}_l)=\Delta(\mathbf{0})$ and the lemma we have

$$\Delta\left(\sum_{i=0}^{l-1}p\cdot|\lambda_i|\cdot\mathbf{x}_i+\sum_{i=l}^{r}p\lambda_i\mathbf{x}_i\right)=\Delta(\mathbf{0}).$$

Thus we get $\Delta(\sum_{i=0}^{r}p\lambda_i\mathbf{x}_i)=\Delta(p\mathbf{z})=\Delta(\mathbf{0})$, which proves the theorem. ∎

Proof of the Lemma. Let $(L_\mu)_{\mu<\xi}$ be the family of all lines in $\mathscr{A}(m^t)$. We shall proceed by induction on μ. Let $N=n_{\xi-\nu}(t, m)$ be very large and suppose Δ: $N^t\to\mathbb{N}$ satisfies the assertion of the lemma for every line L_μ with $\mu<\nu<\xi$. Our object will be to find a homothetic copy $h(n^t)$ of n^t in N^t, where $n=n_{\xi-\nu-1}(t, m)$ is sufficiently large, so that Δ restricted on the set $h(n^t)$ satisfies the assertion of the lemma for every L_μ where $\mu\leqslant\nu$.

Repeating this ξ times we finally obtain a homothetic copy of m^t, $m = n_0(t, m)$, satisfying the lemma. Choose $p = \lfloor N/n \rfloor$ and let Δ^*: $p^{t+1} \to B_{n^t}$ (where B_{n^t} is the n^tth "Bellnumber") be the mapping which associates to every $(t+1)$-tuple $(\mathbf{a}, d) \in p^{t+1}$ the pattern of equivalence on the homothetic copy $\{\mathbf{a} + d\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \in n^t\}$ of n^t. More formally, let $\Delta^*(\mathbf{a}, d) = \Delta^*(\mathbf{a}', d')$ iff $(\Delta(\mathbf{a} + d\boldsymbol{\lambda}) = \Delta(\mathbf{a} + d\boldsymbol{\lambda}')$ iff $\Delta(\mathbf{a}' + d'\boldsymbol{\lambda}) = \Delta(\mathbf{a}' + d'\boldsymbol{\lambda}')$ for every $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}' \in n^t)$. Put $r = n^2$. According to the Gallai–Witt theorem there exists (for N is large enough with respect to n) a homothety $\{(\mathbf{a}, b) + d\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \in r^{t+1}\}$ of r^{t+1} in p^{t+1} on which Δ^* is constant. Thus, the homothetic copies of n^t in N^t given by $\{(\mathbf{a} + d\mathbf{i}) + (b + dj)\,\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \in n^t\}$, where $\mathbf{i} \in r^t$, $j \in r$, have the same pattern with respect to Δ.

Assume that there exist $\mathbf{x}_0, \mathbf{x}_1 \in m^t$ satisfying $\mathbf{x}_1 - \mathbf{x}_0 \in L_\nu$ such that

$$\Delta((\mathbf{a} + d\mathbf{i}) + (b + dj)\,\mathbf{x}_0) = \Delta((\mathbf{a} + d\mathbf{i}) + (b + dj)\,\mathbf{x}_1).$$

Fix $\mathbf{i}_0 \in r^t$ and let $\mathbf{y}_0 = (\mathbf{a} + d\mathbf{i}_0) + b\mathbf{x}_0$ (setting $j = 0$). Denote by $M(\mathbf{y}_0)$ the set of all points in $\mathbf{x}_1$-position with respect to $\mathbf{y}_0$, i.e.,

$$\begin{aligned} M(\mathbf{y}_0) = \{\mathbf{y} \in n^t \mid \exists \mathbf{i} \in r^t, j \in r \text{ such that} & \\ \mathbf{y} &= (\mathbf{a} + d\mathbf{i}) + (b + dj)\,\mathbf{x}_1 \\ \text{and} \quad \mathbf{y}_0 &= (\mathbf{a} + d\mathbf{i}) + (b + dj)\,\mathbf{x}_0\}. \end{aligned}$$

Clearly,

$$\Delta \restriction M(\mathbf{y}_0) = \text{const.} \tag{4}$$

We show that

$$M(\mathbf{y}_0) = \{\mathbf{a} + d\mathbf{i}_0 + b\mathbf{x}_1 + dj(\mathbf{x}_1 - \mathbf{x}_0) \mid j \in r \text{ satisfying } \mathbf{i}_0 - j\mathbf{x}_0 \in r^t\}. \tag{5}$$

If $\mathbf{y} = (\mathbf{a} + d\mathbf{i}) + (b + dj)\,\mathbf{x}_1 \in M(\mathbf{y}_0)$, then

$$\mathbf{y}_0 = (\mathbf{a} + d\mathbf{i}) + (b + dj)\,\mathbf{x}_0 = (\mathbf{a} + d\mathbf{i}_0) + b\mathbf{x}_0.$$

Therefore $\mathbf{i} = \mathbf{i}_0 - j\mathbf{x}_0$. Hence, every $\mathbf{y} \in M(\mathbf{y}_0)$ can be written as

$$\begin{aligned} \mathbf{y} = \mathbf{a} + d\mathbf{i}_0 + b\mathbf{x}_1 + dj(\mathbf{x}_1 - \mathbf{x}_0), & \\ \text{where} \quad j \in r \quad \text{and} \quad \mathbf{i} = \mathbf{i}_0 - j\mathbf{x}_0 \in r^t. & \end{aligned} \tag{6}$$

On the other hand, if $\mathbf{y}'$ satisfies (6) then

$$\mathbf{y}' = \mathbf{a} + d(\mathbf{i}_0 - j\mathbf{x}_0) + (b + dj)\,\mathbf{x}_1$$

and as

$$\mathbf{y}_0 = \mathbf{a} + d(\mathbf{i}_0 - j\mathbf{x}_0) + (b + dj)\,\mathbf{x}_0$$

we infer that $\mathbf{y}' \in M(\mathbf{y}_0)$.

Let $g \in \mathbb{N}$ be such that for any $\mathbf{z}, \mathbf{c} \in m^t$ satisfying $\mathbf{z} - \mathbf{c} = \rho(\mathbf{x}_1 - \mathbf{x}_0)$ for some $\rho \in \mathbb{Q}^+$ we have that $g \cdot \rho \in \mathbb{N}$. For n sufficiently large it follows already that $g \cdot \rho \in n$. Then, in particular, $g \in n$.

We claim that the homothetic copy $h(n^t) = \{(\mathbf{a} + b\mathbf{x}_1 + d\mathbf{s}) + dg\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \in n^t\}$ of n^t in N^t, where $\mathbf{s} = (r - mn, ..., r - mn) \in r^t$, has the property that any two points on a line which is parallel to L_v have the same image with respect to Δ. Let $\mathbf{z}_1 = \mathbf{c} + \rho_1(\mathbf{x}_1 - \mathbf{x}_0)$, $\mathbf{z}_2 = \mathbf{c} + \rho_2(\mathbf{x}_1 - \mathbf{x}_0)$ be two points on a parallel line to L_v in n^t. Without loss of generality we can assume that $\rho_1, \rho_2 \in \mathbb{Q}^+$. Then

$$\begin{aligned} h(\mathbf{z}_i) &= (\mathbf{a} + b\mathbf{x}_1 + d\mathbf{s}) + dg(\mathbf{c} + \rho_i(\mathbf{x}_1 - \mathbf{x}_0)) \\ &= \mathbf{a} + d(\mathbf{s} + g\mathbf{c}) + b\mathbf{x}_1 + d(g\rho_i)(\mathbf{x}_1 - \mathbf{x}_0) \\ &\qquad \text{for} \quad i = 1, 2, \end{aligned}$$

where $\mathbf{s} + g\mathbf{c} \in r^t$ and $(\mathbf{s} + g\mathbf{c}) - (g\rho_i)\,\mathbf{x}_0 \in r^t$. Let $\mathbf{z}_0 = \mathbf{a} + d(\mathbf{s} + g\mathbf{c}) + b\mathbf{x}_0$. Then $\mathbf{z}_1, \mathbf{z}_2 \in M(\mathbf{z}_0)$ and we infer from (4) that $\Delta(h(\mathbf{z}_1)) = \Delta(h(\mathbf{z}_2))$. ∎

3. Concluding Remarks

More generally, h: $\mathbb{R}^t \to \mathbb{R}^t$ is a homothety iff h is of the form $h(\mathbf{b}) = \mathbf{a} + d\mathbf{b}$, where $\mathbf{a} \in \mathbb{R}^t$ and $d \in \mathbb{R} \backslash \{0\}$. Then the following version of the Gallai–Witt theorem is also true (cf. [5, p.38]). For every finite $V \subseteq \mathbb{R}^t$ there exists a finite $W \subseteq \mathbb{R}^t$ such that for every mapping Δ: $W \to \{0, 1\}$ there exists a homothety h: $\mathbb{R}^t \to \mathbb{R}^t$ such that $\Delta(h(\mathbf{b})) = \Delta(h(\mathbf{c}))$ for all $\mathbf{b}, \mathbf{c} \in V$.

Using this result, the same proof as before (with technical modifications concerning the different structure of $S \subseteq \mathbb{R}^t$) can be used to obtain also the following theorem of Spencer. For $S \subseteq \mathbb{R}^t$, S finite, let $\mathscr{A}(S)$ be defined as above with respect to subspaces of $\mathbb{R}^t$.

THEOREM [6]. *Let $S \subseteq \mathbb{R}^t$ be a finite set. Then there exists a finite set $T \subseteq \mathbb{R}^t$ such that for every mapping Δ: $T \to \mathbb{R}$ there exists a homothety h: $\mathbb{R}^t \to \mathbb{R}^t$ and a linear subspace $U \in \mathscr{A}(S)$ with the property $\Delta(h(\mathbf{b})) = \Delta(h(\mathbf{c}))$ iff $\mathbf{b} - \mathbf{c} \in U$ for every $\mathbf{b}, \mathbf{c} \in S$.*

Details are left to the reader.

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